A physical interpretation of the rigidity matrix

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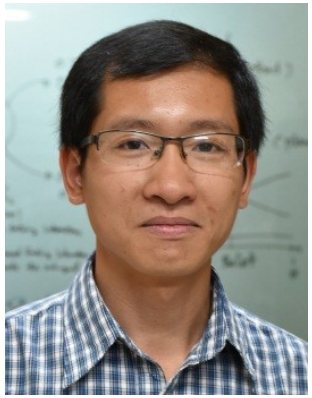
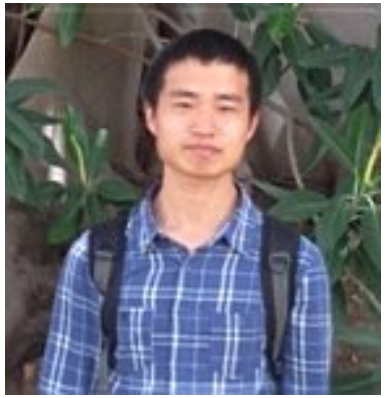
Towards the Discovery of Common Models and Methods

2017

Credits



(A) Minh Hoang Trinh



(B) Zhiyong Sun

(C) Prof. Brian D.O. Anderson

(D) Viet Hoang Pham

CONTENTS

Multi-agent systems & Distributed formation control

Agents and multi-agent systems:

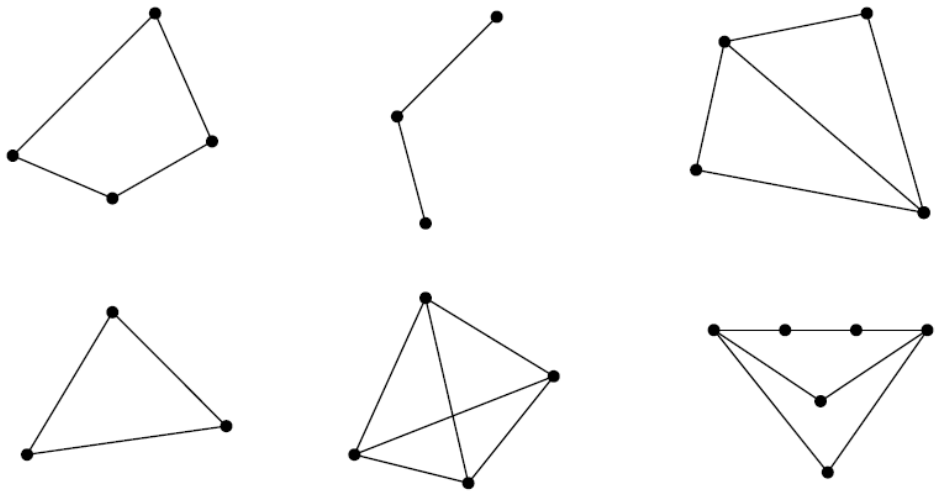
- An agent is understood as a dynamical system.
- A multi-agent system is a collection, a group, or a team of dynamical systems.

Distributed formation control:

- No centralized controller for a given multi-agent system.
- Each agent has its own controller based on interaction with its neighboring agents.
- Only the distances among agents are controlled by relative interactions; $\rightarrow$ but a formation defined w.r.t a global coordinate frame is achieved, upto translations and rotations.

Formation with Distance Constraints

- Only distances (edges) are constrained
- Formation is fixed (rigid) or not-fixed (flex) ?



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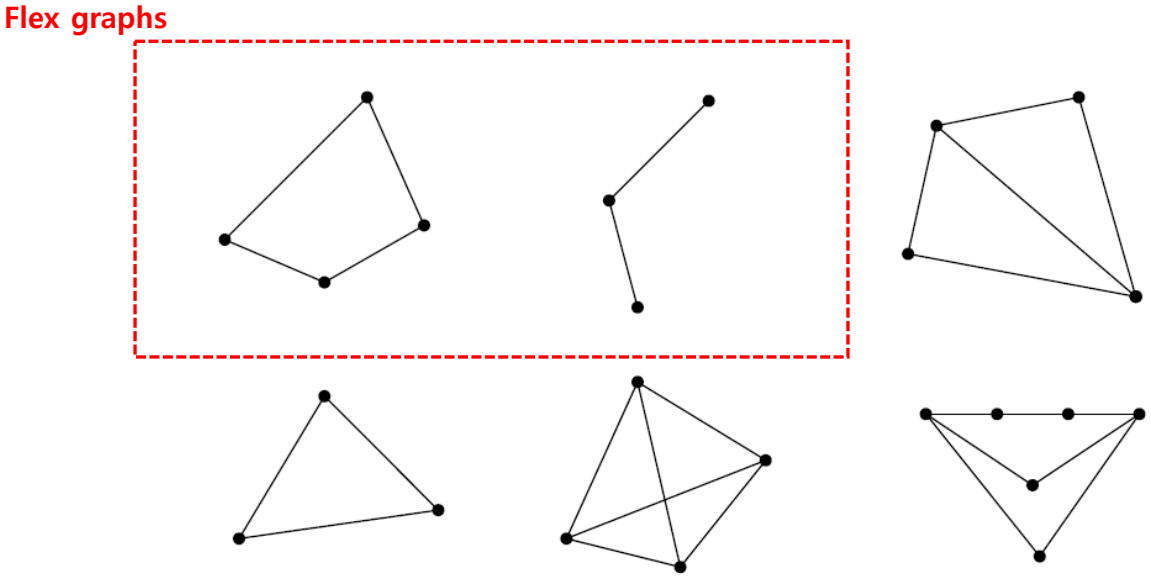

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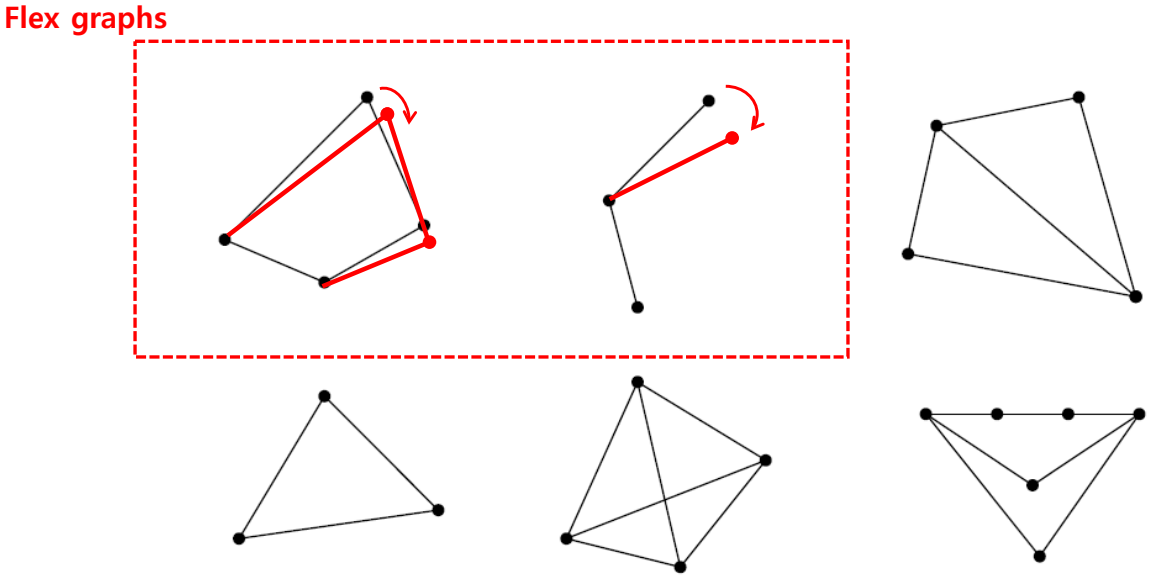

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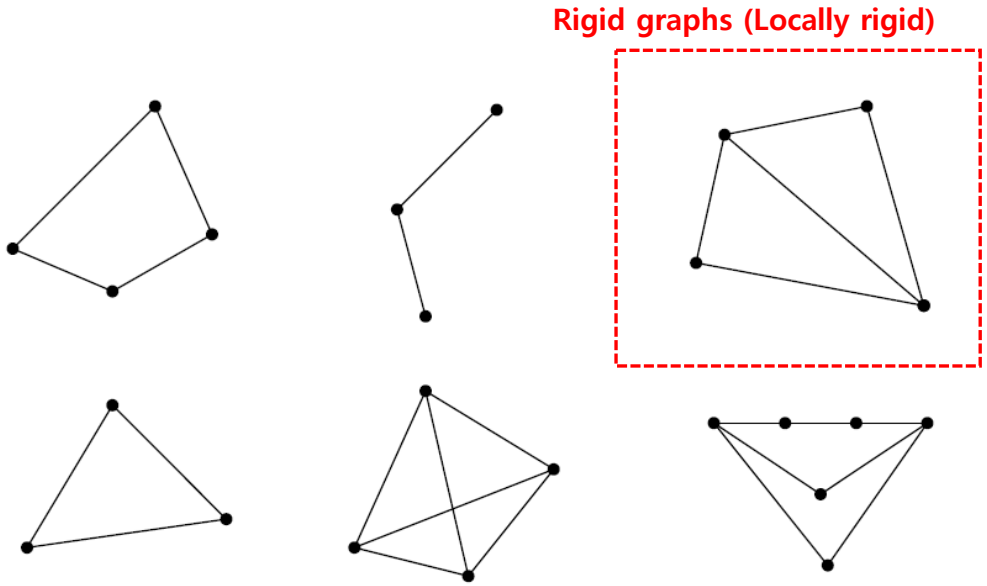

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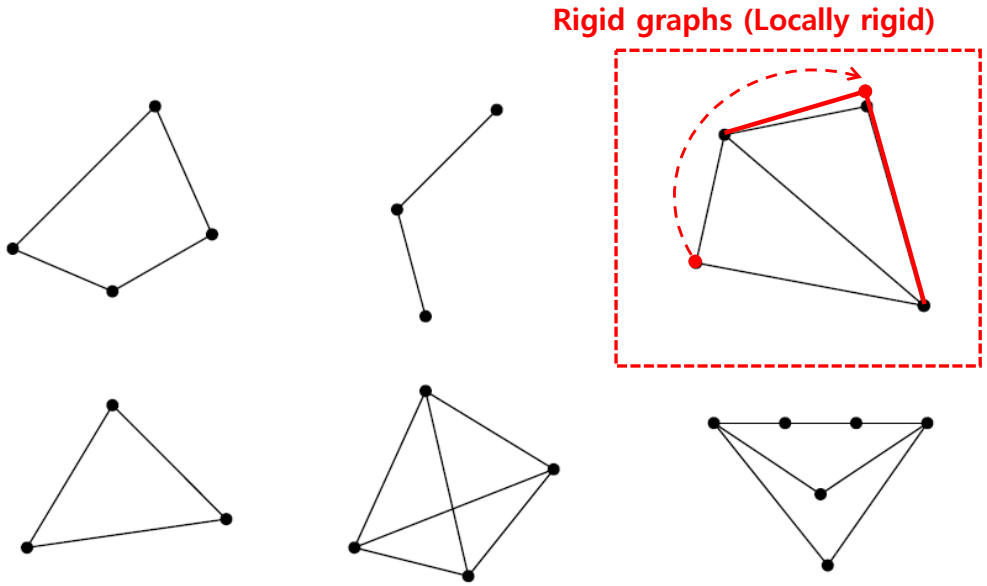

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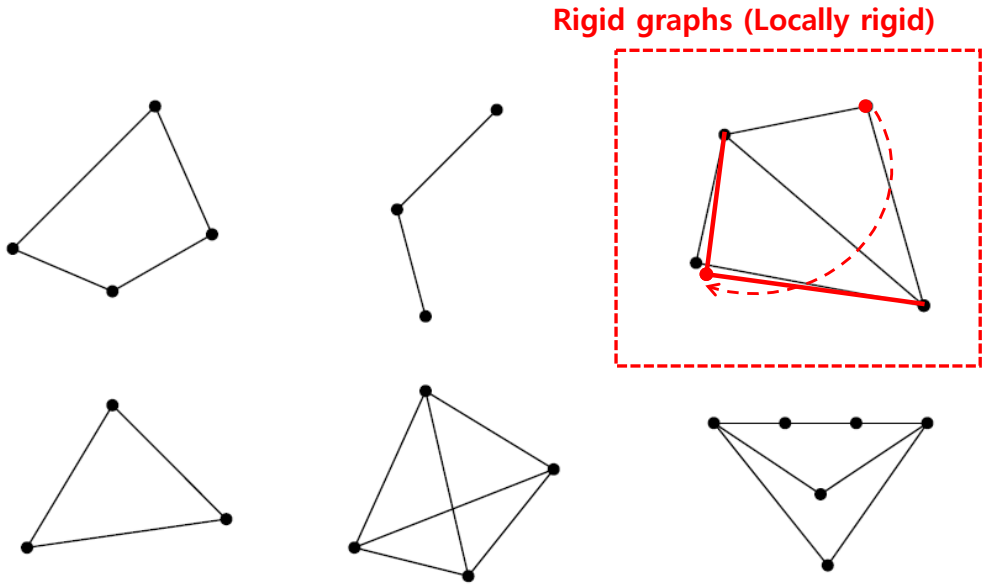

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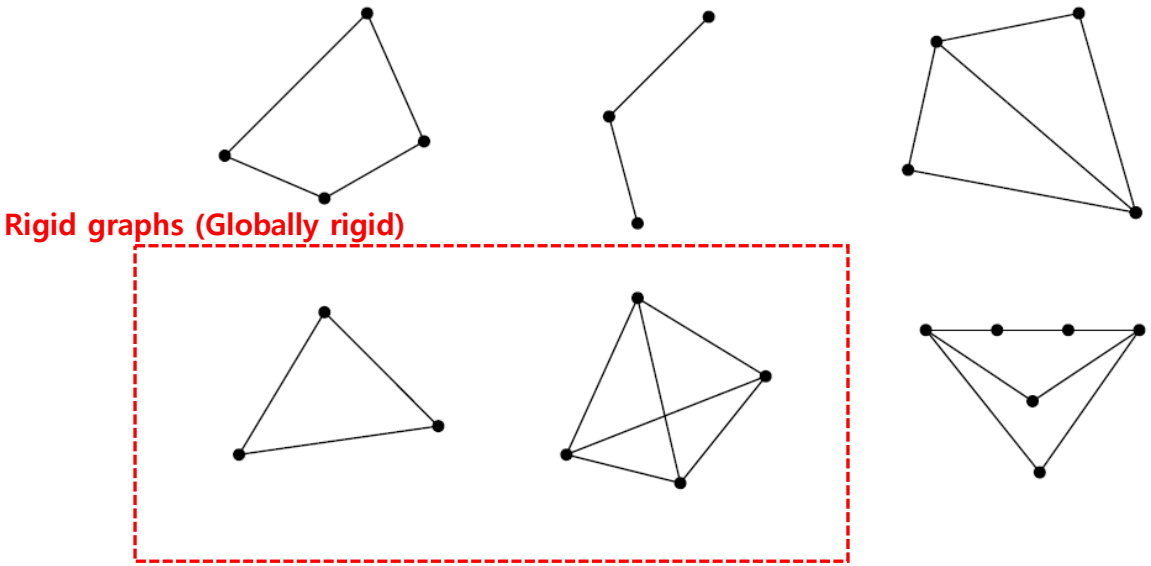

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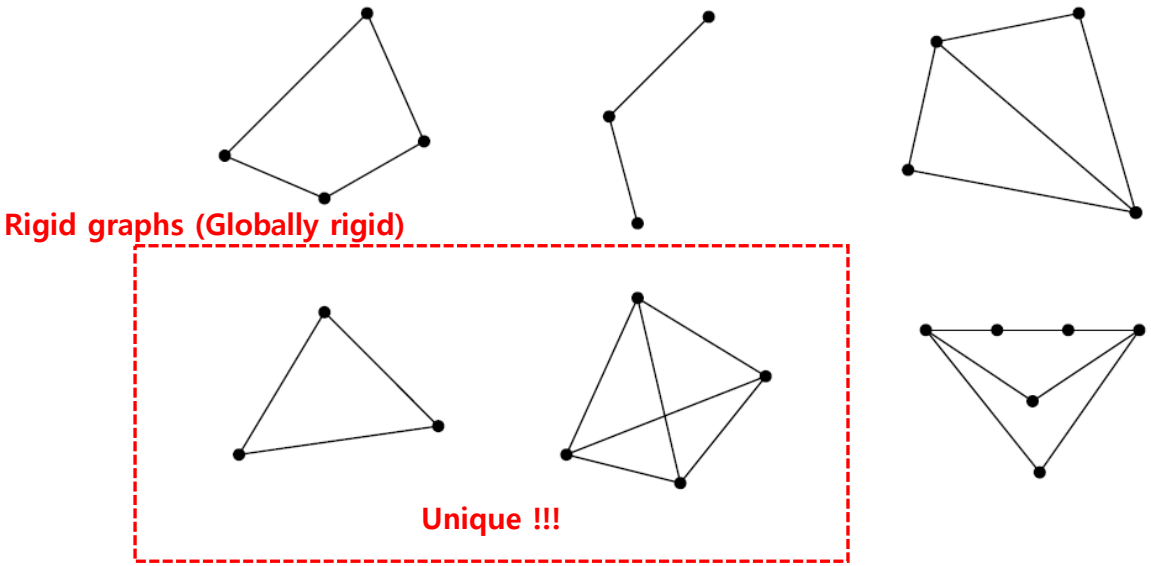

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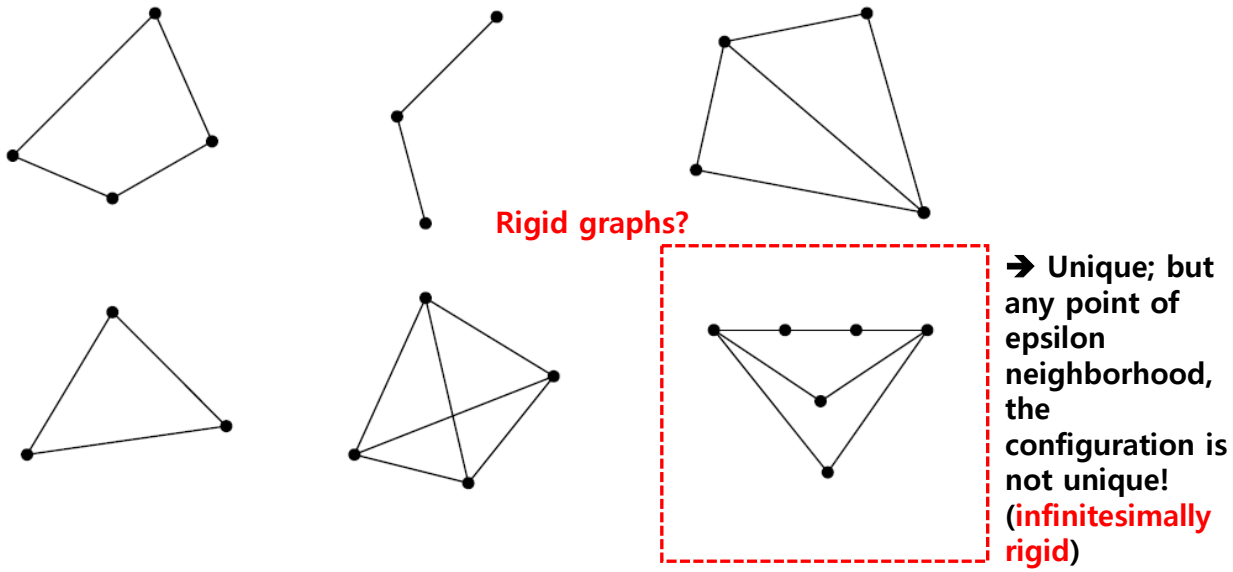
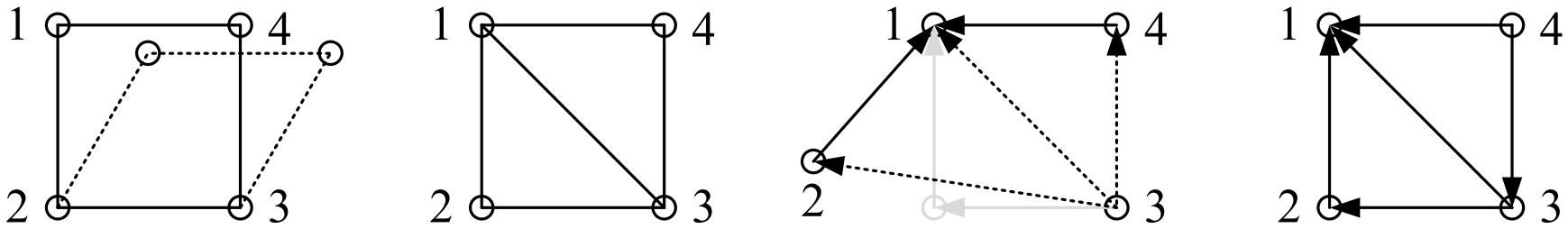


Consistency between the overall and the local tasks

If all the agents complete their local task, then the overall task is achieved? What condition is required for $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in order to satisfy

$$\underbrace{\forall (i,j) \in \mathcal{E}, \|p_i - p_j\| = \|p_i^* - p_j^*\|}_{\text{Equivalence (the local tasks)}} \Rightarrow \underbrace{\forall i,j \in \mathcal{V}, \|p_i - p_j\| = \|p_i^* - p_j^*\|}_{\text{Congruence (the overall task)}}?$$

Rigidity or the persistence of $\mathcal{G}$ specifying the minimum number and the distribution pattern of edges.



(E) Not rigid. (F) Rigid. (G) Not persistent. (H) Persistent.

Use of Graph rigidity for Consistency in Task

Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$, let us assign $p_i \in \mathbb{R}^n$ to each vertex i for all $i \in \mathcal{V}$.

- Realization: $p = (p_1^T, \ldots, p_N^T)^T \in \mathbb{R}^{nN}$, Framework: $(\mathcal{G}, p)$
- Equivalence: Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are equivalent if

$$\forall\, (i,j) \in \mathcal{E},\ \|p_i - p_j\| = \|q_i - q_j\|.$$

- Congruence: Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, q)$ are congruent if

$$\forall\, i,j \in \mathcal{V},\ \|p_i - p_j\| = \|q_i - q_j\|.$$

Definition (Rigidity)

A framework $(\mathcal{G}, p)$ is rigid if there exists a neighborhood U_p of p such that all frameworks equivalent to $(\mathcal{G}, p)$ are congruent in U_p.

☞ If $(\mathcal{G}, p)$ is rigid, then the overall task and the local tasks is consistent.

CONTENTS

Incidence rigidity & Similarity

- $\mathcal{F} = (\mathcal{G}, p)$: a framework in $\mathbb{R}^2$
  - $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1, \ldots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, $|\mathcal{V}| = n, |\mathcal{E}| = m$
  - $p_i = [x_i, y_i]^T$, $p = [p_1^T, \ldots, p_n^T]^T$: a realization in $\mathbb{R}^2$
  - $H \in \mathbb{R}^{m \times n}$: the incidence matrix

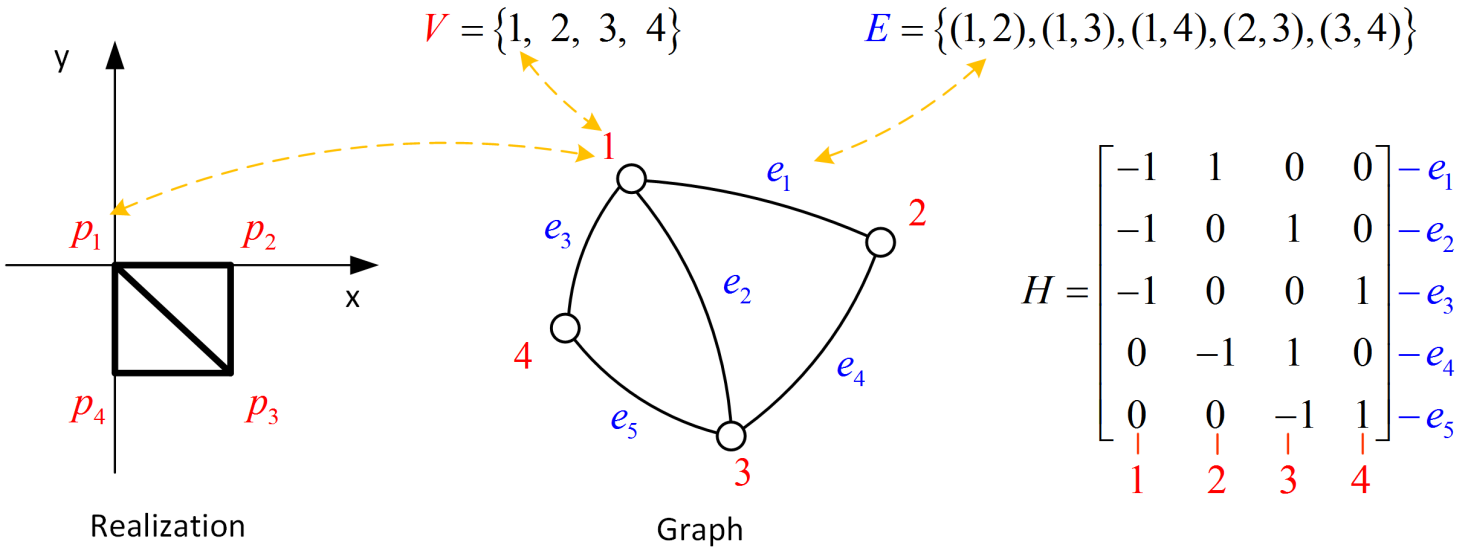


Figure: A framework with four vertices and five edges.

Incidence rigidity & Similarity

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  - $p_i = [x_i, y_i]^T$, $p = [p_1^T, \ldots, p_n^T]^T$: a realization in $\mathbb{R}^2$

- $\mathcal{F} = (\mathcal{G}, p)$ and $\mathcal{F}' = (\mathcal{G}, p')$ are similar iff. $\exists \zeta > 0$ such that

$$\|p_i - p_j\| = \zeta \|p'_i - p'_j\|, \quad \forall i, j \in \mathcal{V} \tag{1}$$

ζ: *the scale factor*

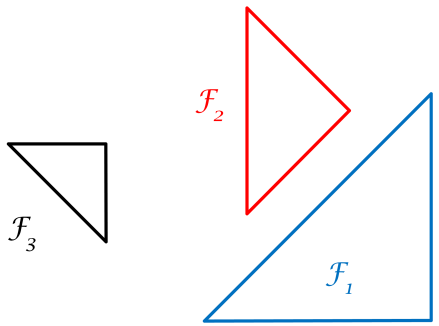

Figure: Similar frameworks: $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.

RIGIDITY MATRIX

- Denote $z_{ij} = p_j - p_i$, $(i,j) \in \mathcal{E}$: displacement vector
- Labeling m edges, we have $z = [z_1^T, \ldots, z_m^T]^T = (H \otimes I_2)p \in \mathbb{R}^{2m}$

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- The distance function $f_{\mathcal{G}} : \mathbb{R}^{2n} \to \mathbb{R}^m$,

$$f_{\mathcal{G}} : p \mapsto \left[\|z_1\|^2, \ldots, \|z_m\|^2\right]^T.$$

- The rigidity matrix $R := \frac{1}{2}\partial f_{\mathcal{G}}(p)/\partial p \in \mathbb{R}^{m \times 2n}$

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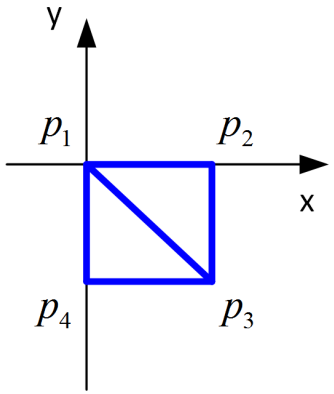

$$R = \begin{bmatrix} (p_1-p_2)^T & -(p_1-p_2)^T & 0 & 0 \\ (p_1-p_3)^T & 0 & -(p_1-p_3)^T & 0 \\ (p_1-p_4)^T & 0 & 0 & -(p_1-p_4)^T \\ 0 & (p_2-p_3)^T & -(p_2-p_3)^T & 0 \\ 0 & 0 & (p_3-p_4)^T & -(p_3-p_4)^T \end{bmatrix}$$

FIGURE: The rigidity matrix R.

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- The rigidity matrix $R := \frac{1}{2}\partial f_{\mathcal{G}}(p)/\partial p \in \mathbb{R}^{m \times 2n}$
  - A framework is *infinitesimally rigid* iff. $rank(R) = 2n - 3$

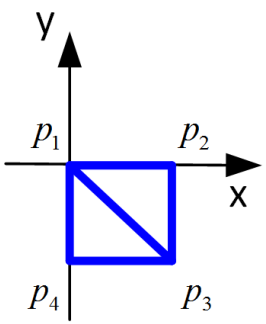
CONTENTS

THE SYMMETRIC RIGIDITY MATRIX

The *symmetric rigidity matrix*[1]

$$M := R^T R. \tag{2}$$

where R is the rigidity matrix.



$$M = R^T R = \begin{bmatrix} \sum_{j=2,3,4} (p_1 - p_j)(p_1 - p_j)^T & -(p_1 - p_2)(p_1 - p_2)^T & -(p_1 - p_3)(p_1 - p_3)^T & -(p_1 - p_4)(p_1 - p_4)^T \\ -(p_2 - p_1)(p_2 - p_1)^T & \sum_{j=1,3} (p_2 - p_j)(p_2 - p_j)^T & -(p_2 - p_3)(p_2 - p_3)^T & 0 \\ -(p_3 - p_1)(p_3 - p_1)^T & -(p_3 - p_2)(p_3 - p_2)^T & \sum_{j=1,2,4} (p_3 - p_j)(p_3 - p_j)^T & -(p_3 - p_4)(p_3 - p_4)^T \\ -(p_4 - p_1)(p_4 - p_1)^T & 0 & -(p_4 - p_3)(p_4 - p_3)^T & \sum_{j=1,3} (p_4 - p_j)(p_4 - p_j)^T \end{bmatrix}$$

FIGURE: The symmetric rigidity matrix M resembles the graph Laplacian matrix.

[1] **Zelazo2012**.

THE SYMMETRIC RIGIDITY MATRIX

The *symmetric rigidity matrix*[1]

$$M := R^T R. \tag{2}$$

where R is the rigidity matrix. Some properties of M:

- $M \in \mathbb{R}^{2n \times 2n}$ is symmetric, positive semidefinite.
- $\mathcal{N}(M) = \mathcal{N}(R)$, $rank(M) \leq 2n - 3$.
  $\Rightarrow$ M has at least three zero eigenvalues.
- $\mathcal{F}$ is infinitesimally rigid iff. $rank(M) = 2n - 3$.

[1] **Zelazo2012**.

EIGENVALUES AND EIGENVECTORS OF M

- The symmetric rigidity matrix M has
  - Eigenvalues: $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$
  - Eigenvectors: $v^1, v^2, \ldots, v^{2n}$, where $v^k = [(v_1^k)^T, \ldots, (v_n^k)^T]^T \in \mathbb{R}^{2n}$

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- v^1, v^2, v^3 associate with three zero eigenvalues[2]

$$
\begin{aligned}
v^1 &= [1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0]^T, \\
v^2 &= [0 \quad 1 \quad 0 \quad 1 \quad \ldots \quad 0 \quad 1]^T, \\
v^3 &= [-y_1 \quad x_1 \quad -y_2 \quad x_2 \quad \ldots \quad -y_n \quad x_n]^T.
\end{aligned}
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[2] **Sun2015a**.

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v^2 &= [0 \quad 1 \quad 0 \quad 1 \quad \ldots \quad 0 \quad 1]^T, \\
v^{3'} &= v^3 + \bar{y} v^1 - \bar{x} v^2,
\end{aligned}
$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$.

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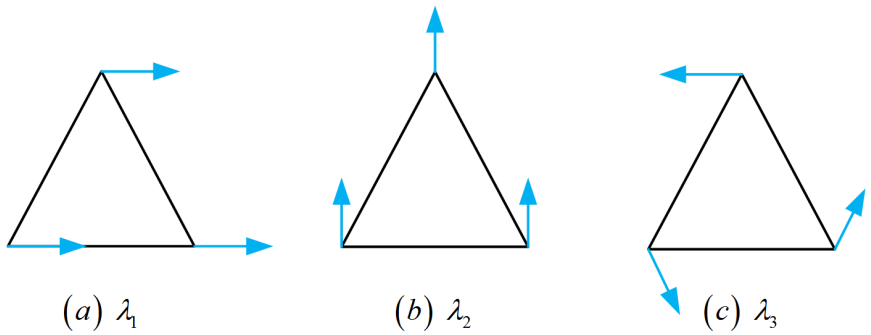

FIGURE: $v^1, v^2, v^{3'}$ correspond to infinitesimally rigid motions.

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- For $\lambda_k \neq 0 \Rightarrow$ physical interpretation of v^k?

STATICS OF FRAMEWORKS

Framework: rods and joints model[2]

[2] **Roth1981**.

STATICS OF FRAMEWORKS

Framework: rods and joints model[2]

- Stress: a set of scalars $w = [w_{ij}]_{(i,j)\in\mathcal{E}}$ defined for each edge
- Equilibrium stress:

$$\sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \quad \forall i = 1, \ldots, n. \tag{3}$$

- A stress is *trivial* when $w_{ij} = 0, \forall (i,j) \in \mathcal{E}$.
- Stress free: Only the trivial makes the equilibrium stress be satisfied
- Rigid graphs: Stress free
- Minimally rigid graphs: Stress free & All edges linearly independent (in the sense of rigidity matrix)
- Flex graphs: Edges linearly dependent (i.e., non-trivial w_{ij} makes the equilibrium stress be satisfied)

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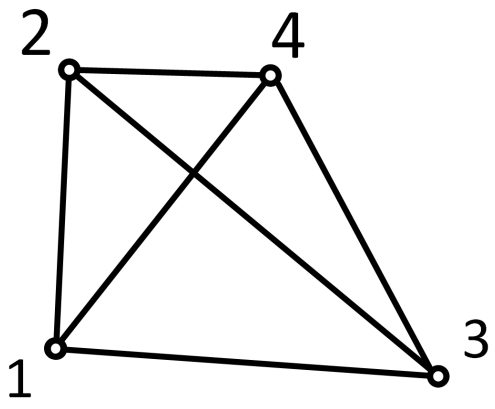
Framework: rods and joints model[3]



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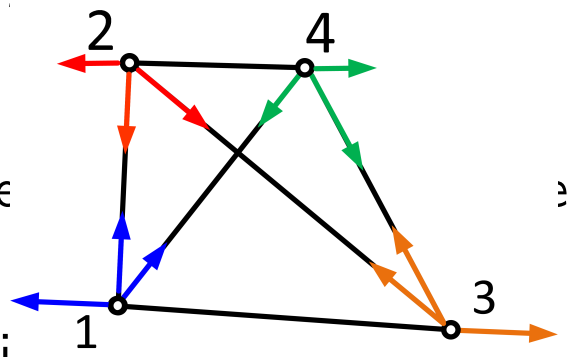

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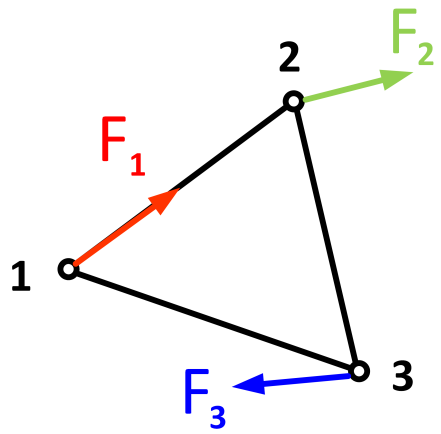
STATICS OF FRAMEWORKS

- $F = [F_1^T, \ldots, F_n^T]^T \in \mathbb{R}^{2n}$ is an *equilibrium force* if

$$\sum_{i=1}^{n} F_i = \mathbf{0}, \tag{5}$$

$$\sum_{i=1}^{n} p_i \times F_i = \mathbf{0}, \tag{6}$$

where $\times$ denotes the cross product.



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  where $\times$ denotes the cross product.

- F is *resolvable* if $\exists$ scalars w_{ij} s.t.

$$F_i + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \tag{7}$$

  for all $i = 1, \ldots, n$.

PHYSICAL MEANING OF EIGENVECTORS OF M

- For each eigenvector $v^k = [v_1^{kT}, \ldots, v_n^{kT}]^T$,

$$Mv^k = R^T R v^k = \lambda_k v^k$$

- Let $w = [w_{ij}]_{(i,j)\in\mathcal{E}} = Rv^k \in \mathbb{R}^m$,

$$Mv^k = R^T(Rv^k) = R^T w = \lambda_k v^k, \quad k = 1, \ldots, 2n. \tag{8}$$

$$\text{or} - \lambda_k v_i^k + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = 0, \quad i = 1, \ldots, n; k = 1, \ldots, 2n. \tag{9}$$

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- Let $w = [w_{ij}]_{(i,j)\in\mathcal{E}} = Rv^k \in \mathbb{R}^m$,

$$Mv^k = R^T(Rv^k) = R^T w = \lambda_k v^k, \quad k = 1, \ldots, 2n. \tag{8}$$

$$\text{or} - \lambda_k v_i^k + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = 0, \quad i = 1, \ldots, n; k = 1, \ldots, 2n. \tag{9}$$

THEOREM

Given an infinitesimally rigid framework $\mathcal{F}$ in a plane. Then each vector $F = -\lambda_k v^k$ $(k = 4, \ldots, 2n-3)$ is a resolvable force, where v^k is the eigenvector corresponding to a nonzero eigenvalue of the symmetric rigidity matrix M of $\mathcal{F}$.

PHYSICAL MEANING OF EIGENVECTORS OF M

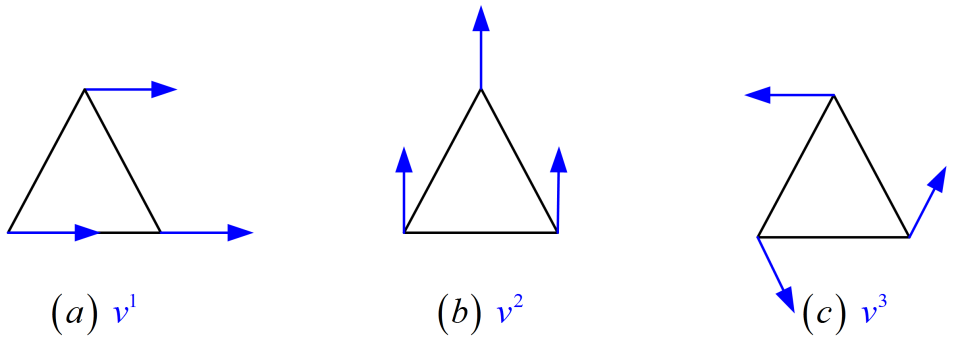
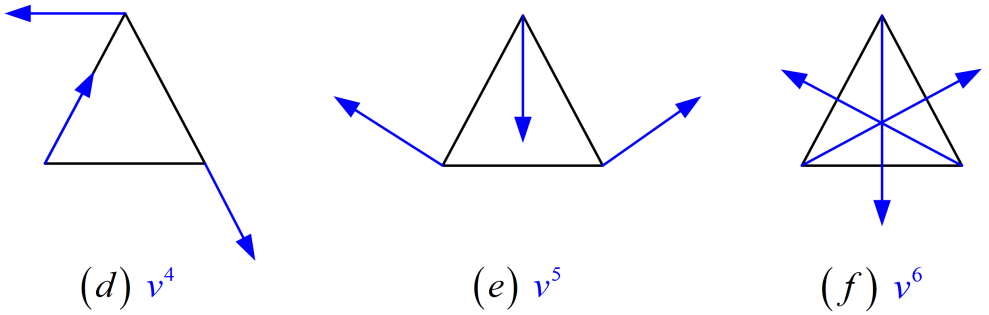
Physical interpretations	**Example: equilateral triangle framework**
v^1, v^2, v^3 • Infinitesimally rigid motions • Trivial stresses	
$v^4, v^5, \ldots, v^{2n}$ • Equilibrium forces & Resolvable forces	

FIGURE: The eigenvectors of an equilateral triangular frameworks

FURTHER PROPERTIES OF EIGENVALUES OF MATRIX M

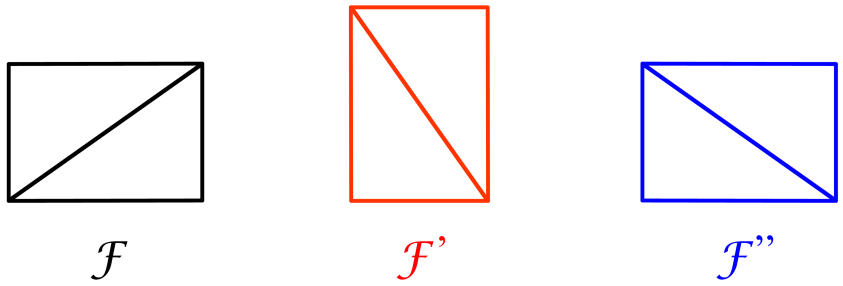
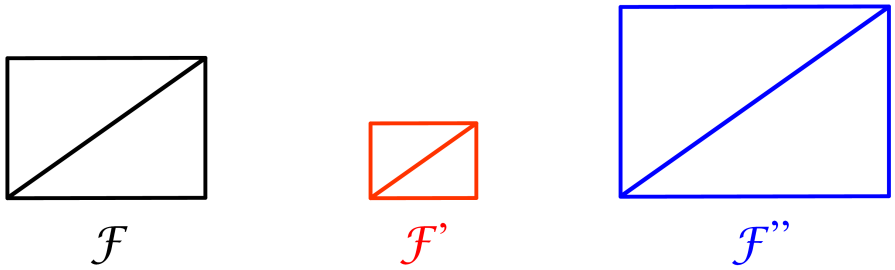
Properties	**Examples**
Invariant to • translation • rotation • reflection $\lambda_i = \lambda_i^{'},\ \forall i = 1,\ldots,2n.$	
Proportional to • (scale factor)^2 $\lambda_i = \zeta^2 \lambda_i^{'},\ \forall i = 1,\ldots,2n.$	
Increasing when adding more edges $\lambda_i \leq \lambda_i^{'},\ \forall i = 1,\ldots,2n.$	

FIGURE: $\mathcal{F}$ has matrix M with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, and $\mathcal{F}'$ has matrix M' with eigenvalues $\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_{2n}$.

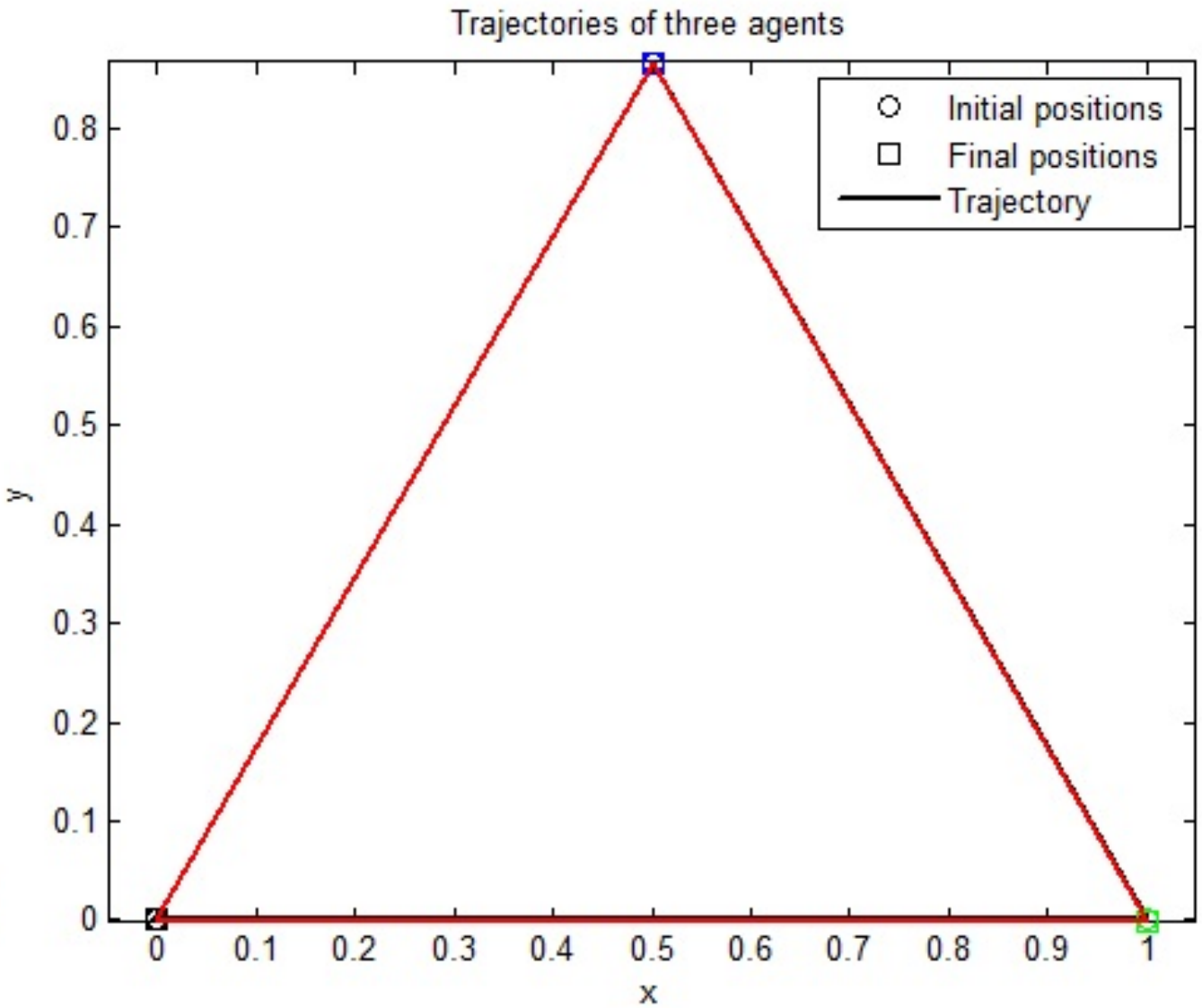
CONTENTS

Simulation

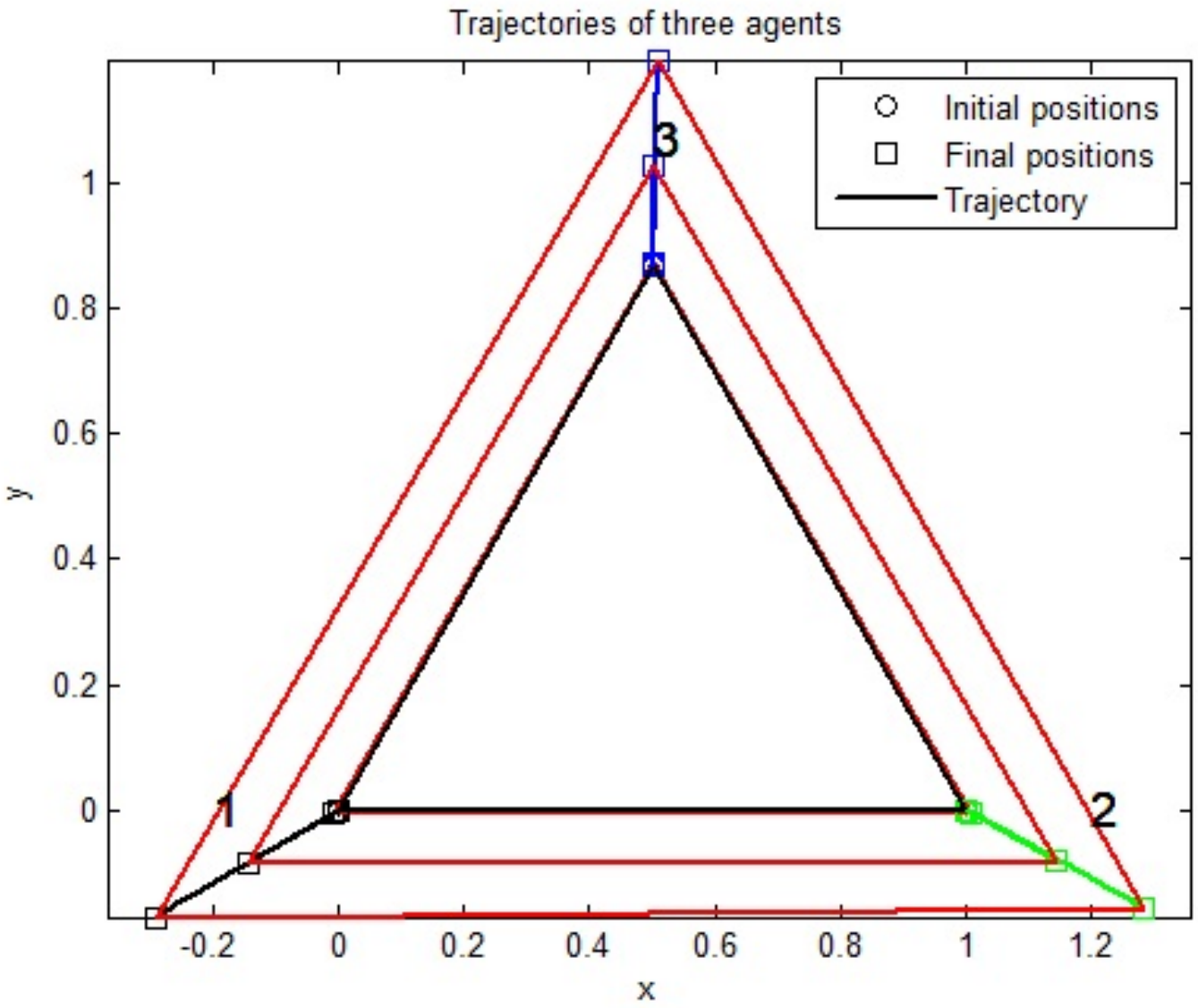
- Consider the triangular framework depicted in the case (d) in page 13 (i.e., $k = 4$).
  - Stress forces along the edges $\sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$
  - Forces to each nodes $-\lambda_4 v_i^4$
- Each node is governed by double integrator dynamics (i.e., $\ddot{p}_i = u_i$).

1) $u_i = -\lambda_4 v_i^4 + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$
2) $u_i = -\lambda_4 v_i^4(0) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$
3) $u_i = -\lambda_4 normalized(v_i^4(0) + 0.1 q_i^4(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$, where $(v_i^4)^T q_i^4 = 0$
4) $u_i = -\lambda_4 normalized(v_i^4(0) + 0.5 q_i^4(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$
5) $u_i = -\lambda_4 normalized(v_i^4(0) + 1.0 q_i^4(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$

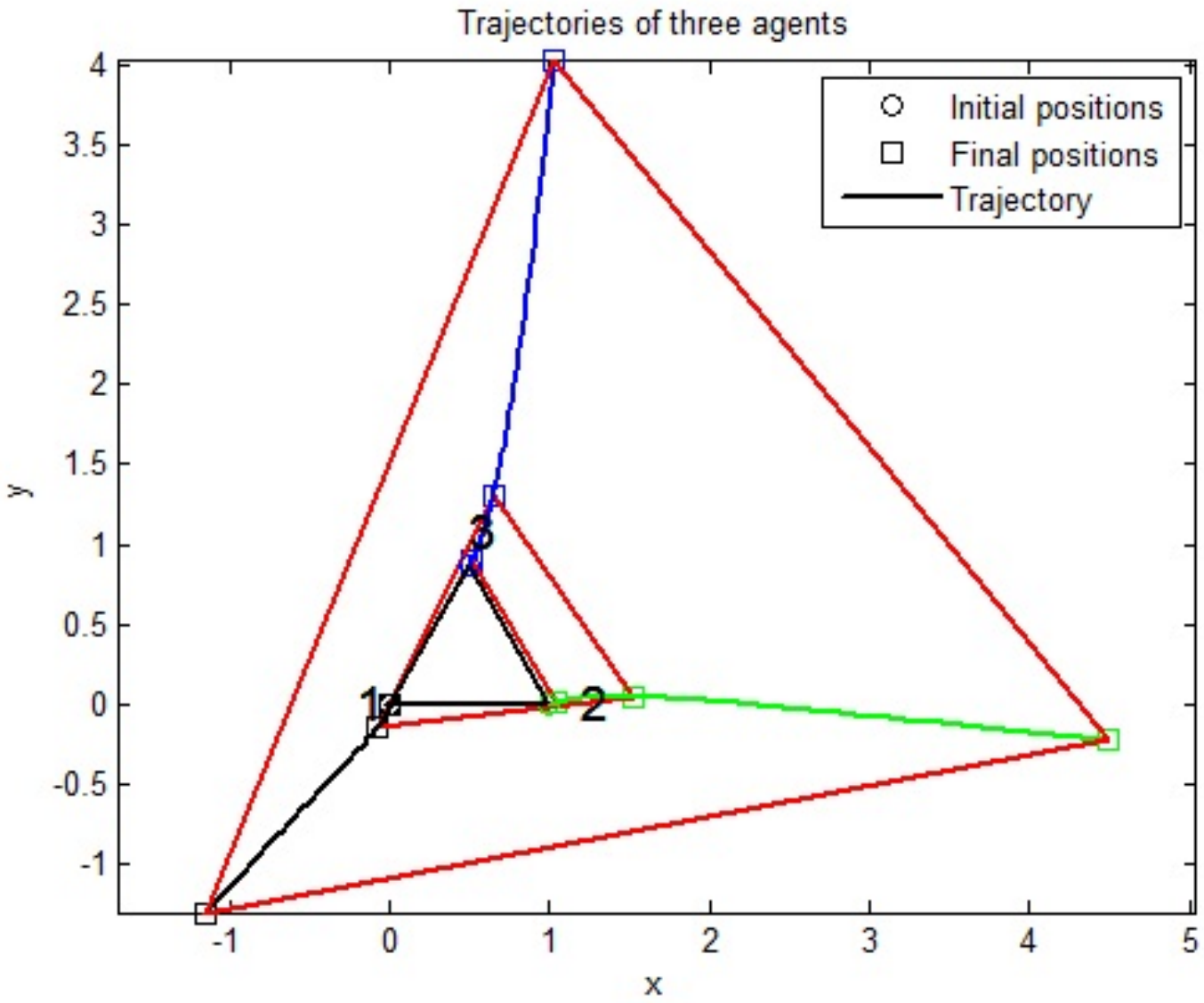
Case 1


Trajectories of three agents
Initial positions
Final positions
Trajectory
y
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

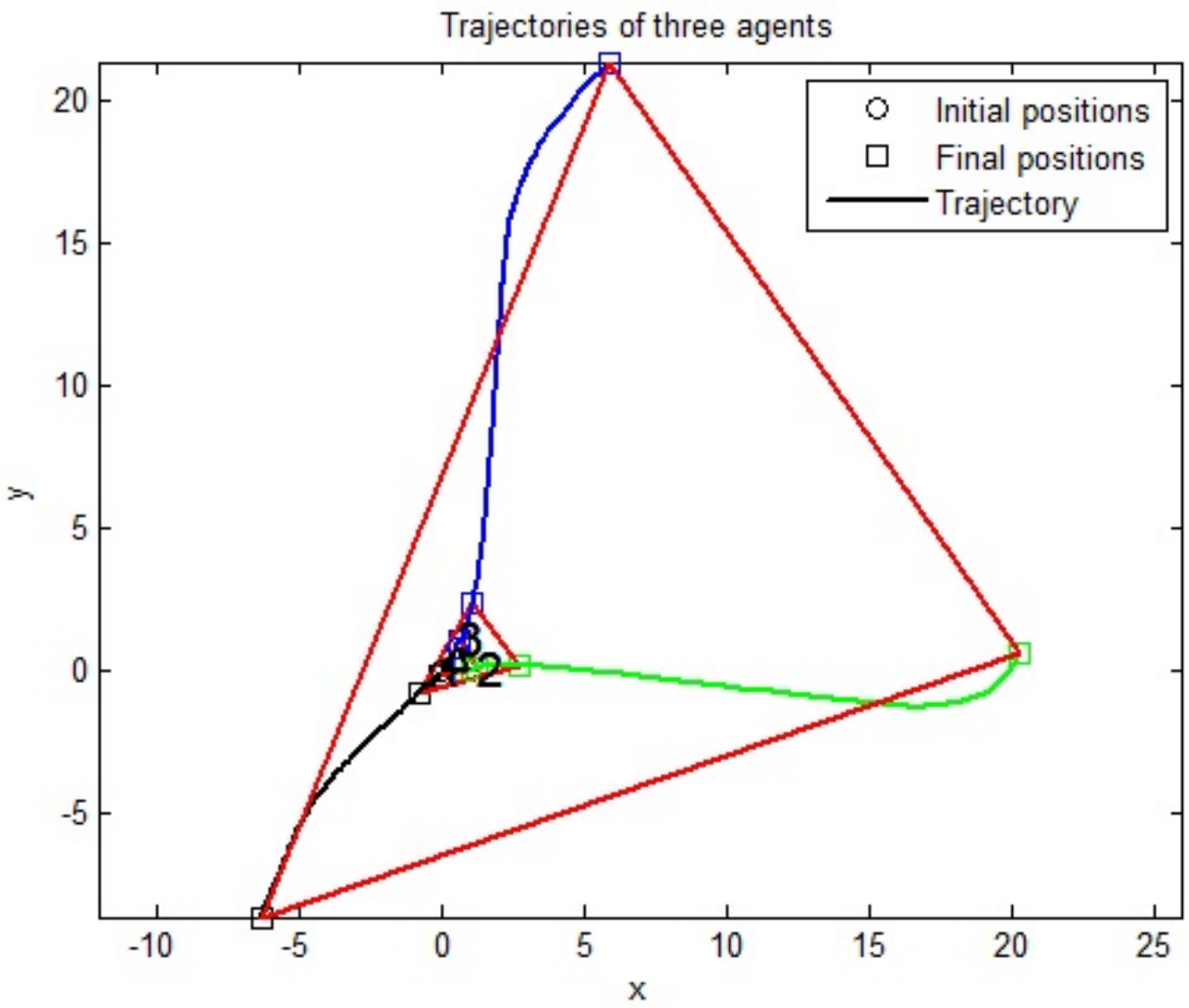
Case 2


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
y
x
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2

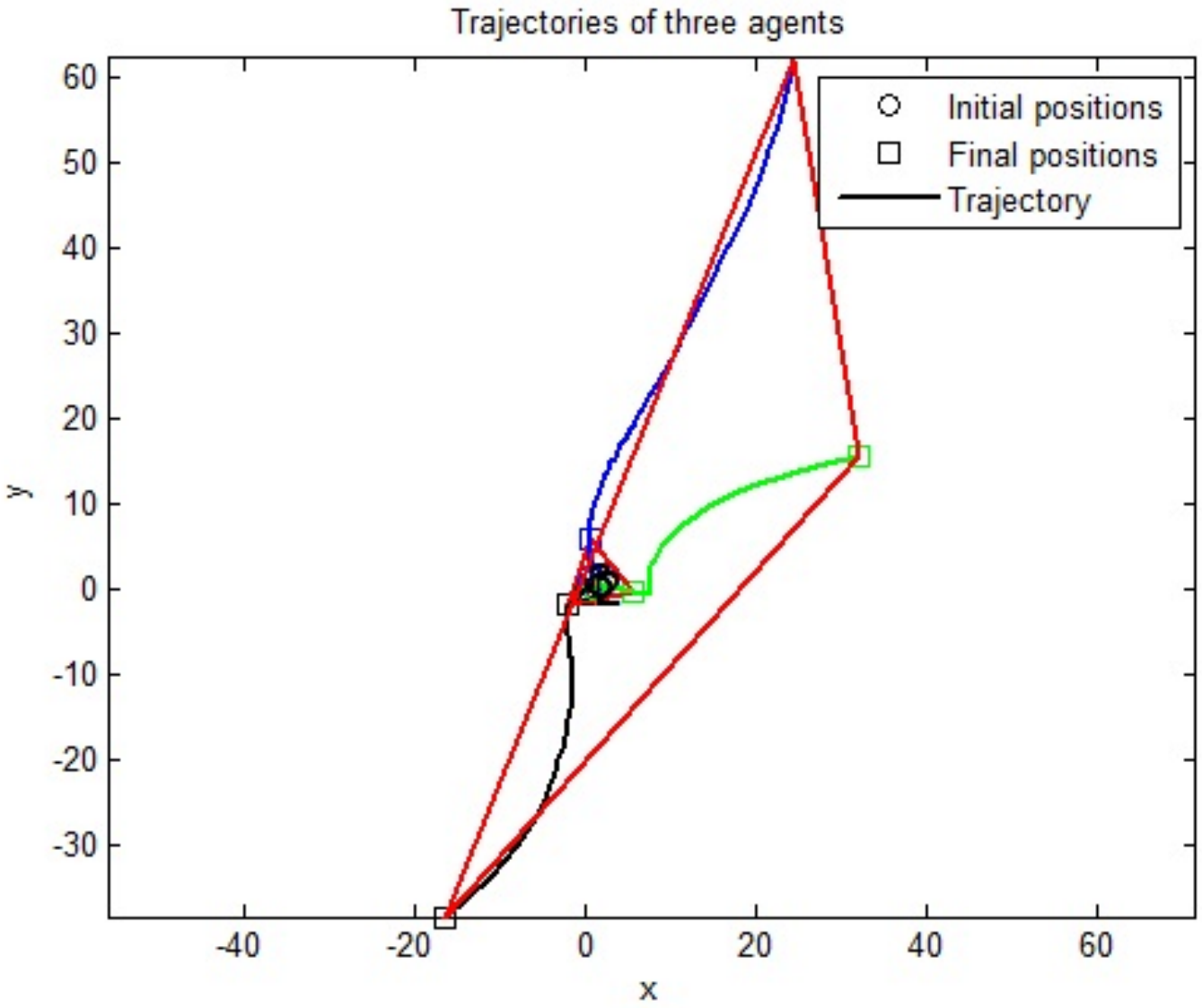
Case 3


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
x
y
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
0
1
2
3
4
5

Case 4


Trajectories of three agents
Initial positions
Final positions
Trajectory
y
x
20
15
10
5
0
-5
-10
-5
0
5
10
15
20
25

Case 5

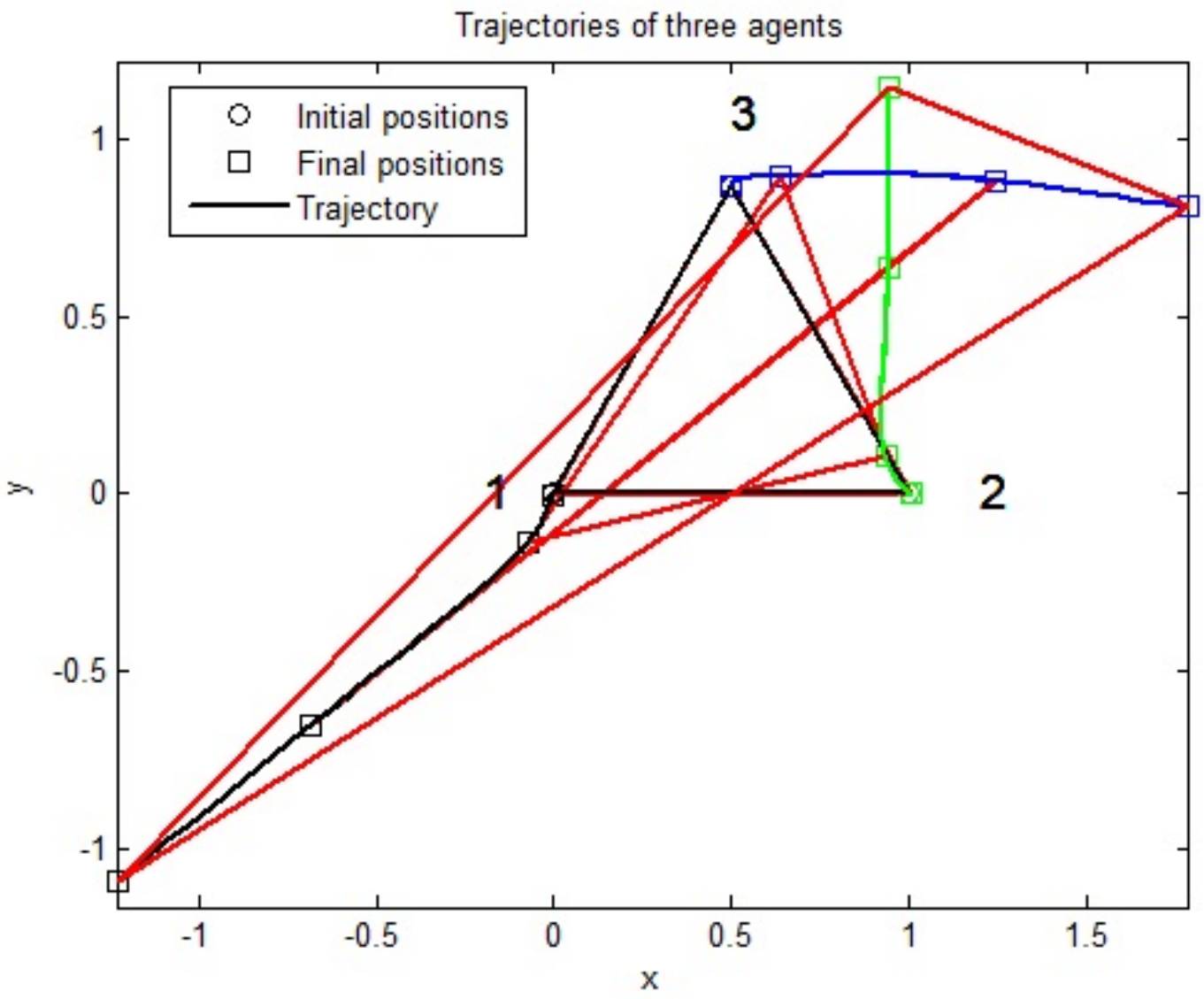

Trajectories of three agents
Initial positions
Final positions
Trajectory
60
50
40
30
20
10
0
-10
-20
-30
y
-40
-20
0
20
40
60
x

Simulation (Cont.)

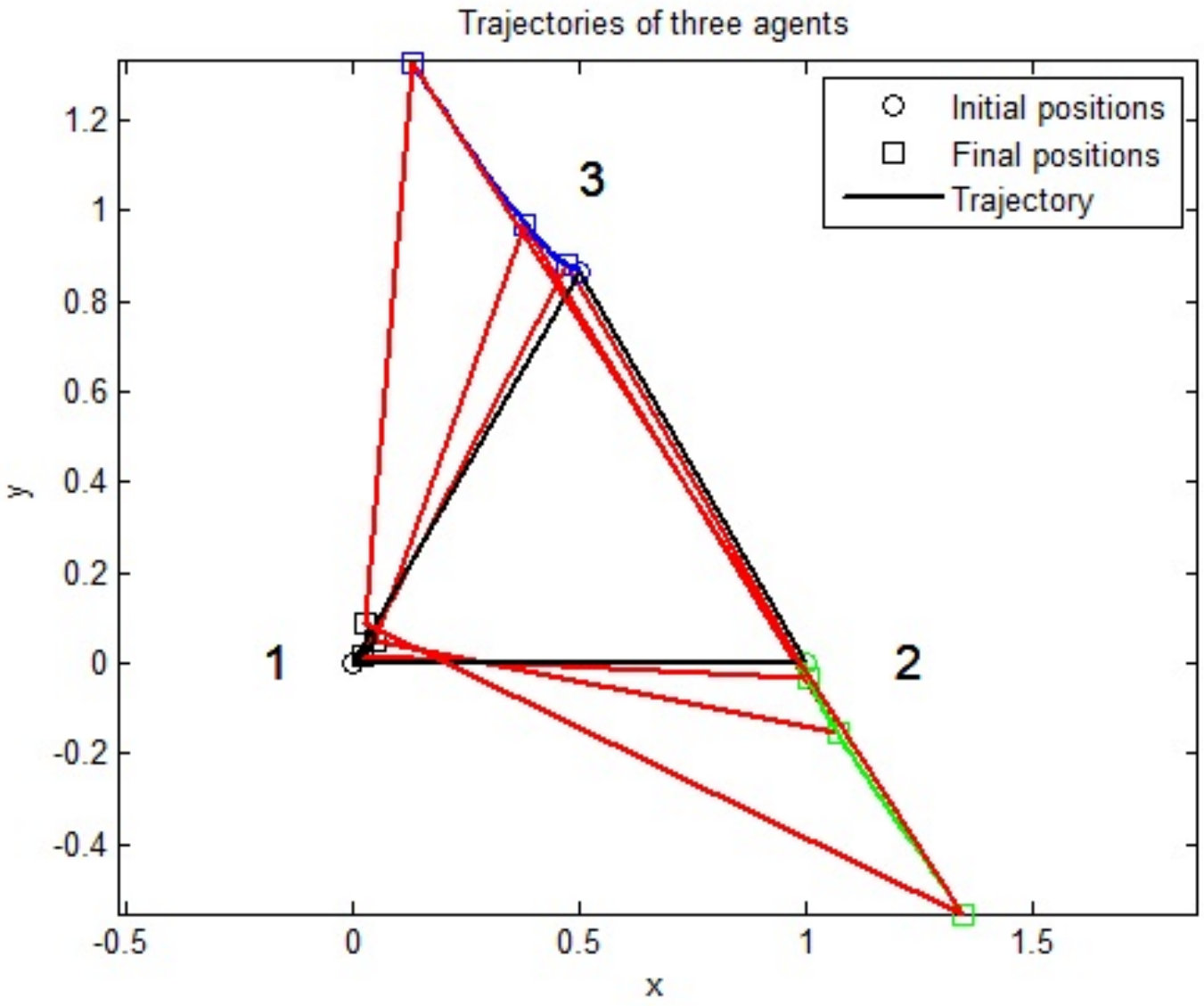
- Each node is governed by double integrator dynamics (i.e., $\ddot{p}_i = u_i$).

6) $u_i = -\lambda_4(0)v_i^4(0) + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j)$
7) $u_i = -\lambda_4(0)v_i^4(0) + \sum_{j\in\mathcal{N}_i} w_{ij}(0)(p_i - p_j)$
8) $u_i = -\lambda_4(0)normalized(v_i^4(0) + 0.1q_i^4(0)) + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j)$, where $(v_i^4)^T q_i^4 = 0$
9) $u_i = -\lambda_4(0)normalized(v_i^4(0) + 0.5q_i^4(0)) + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j)$
10) $u_i = -\lambda_4(0)normalized(v_i^4(0) + 1.0q_i^4(0)) + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j)$

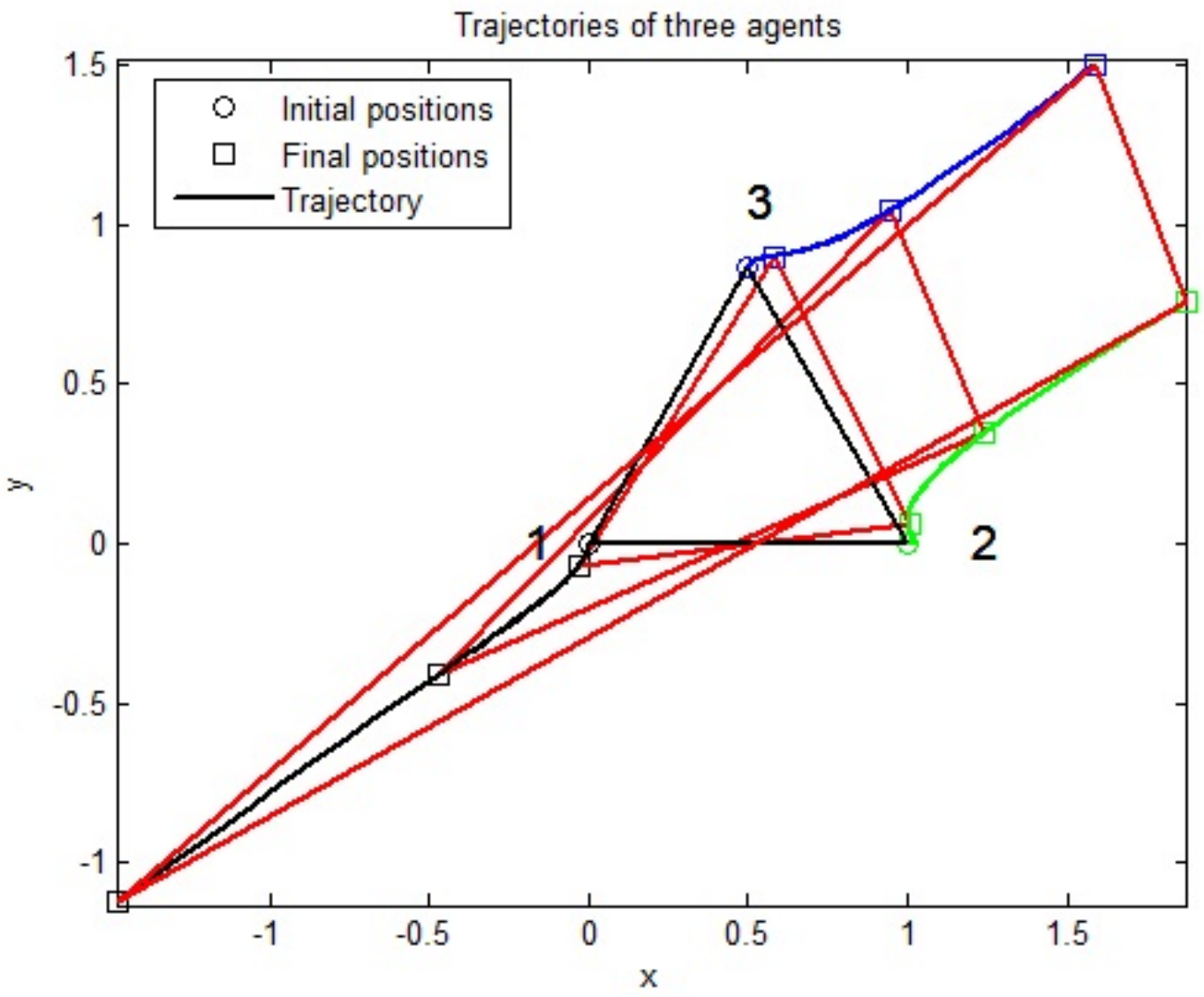
Case 6


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
1.5
x

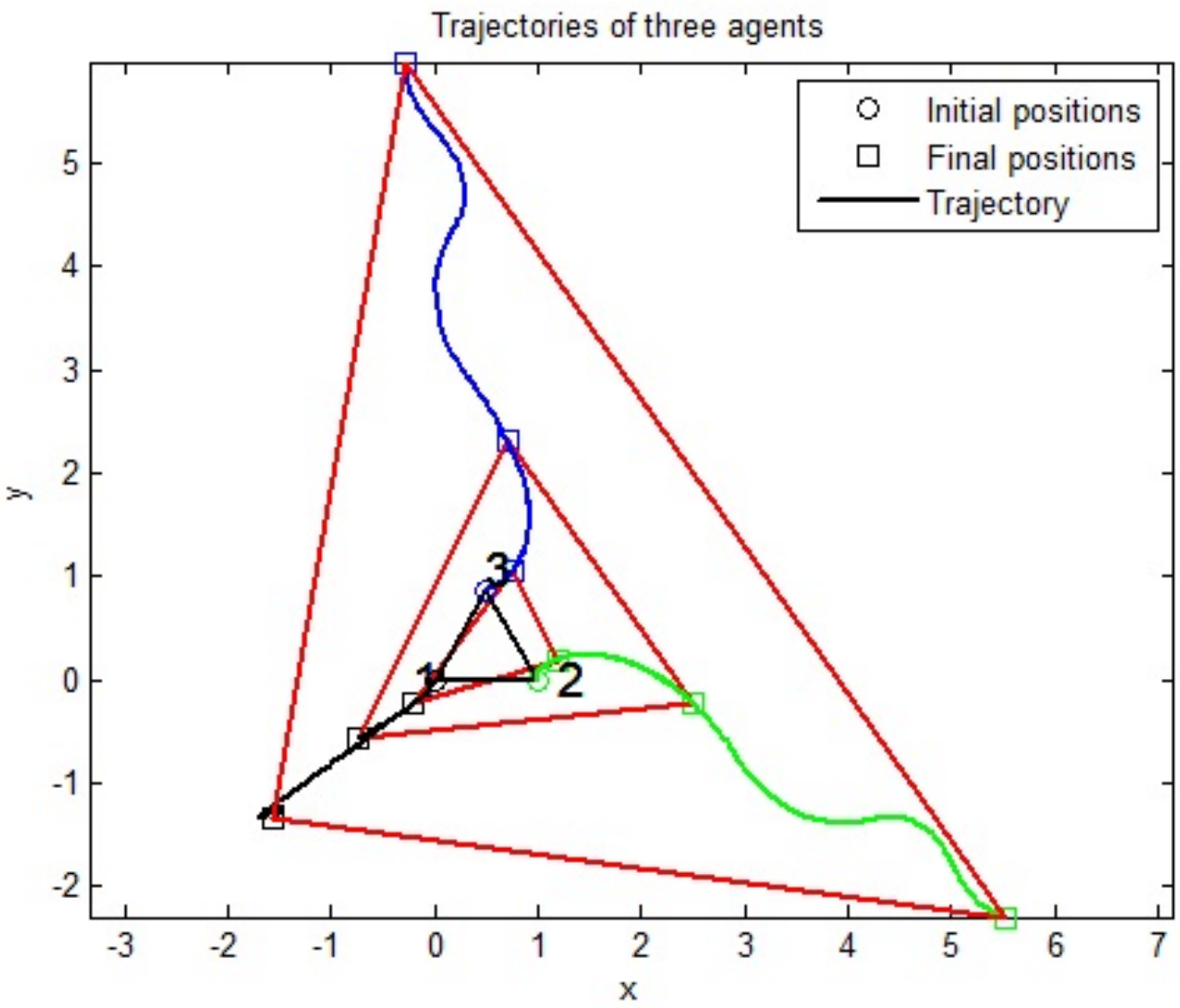
Case 7


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
y
-0.5
0
0.5
1
1.5
x

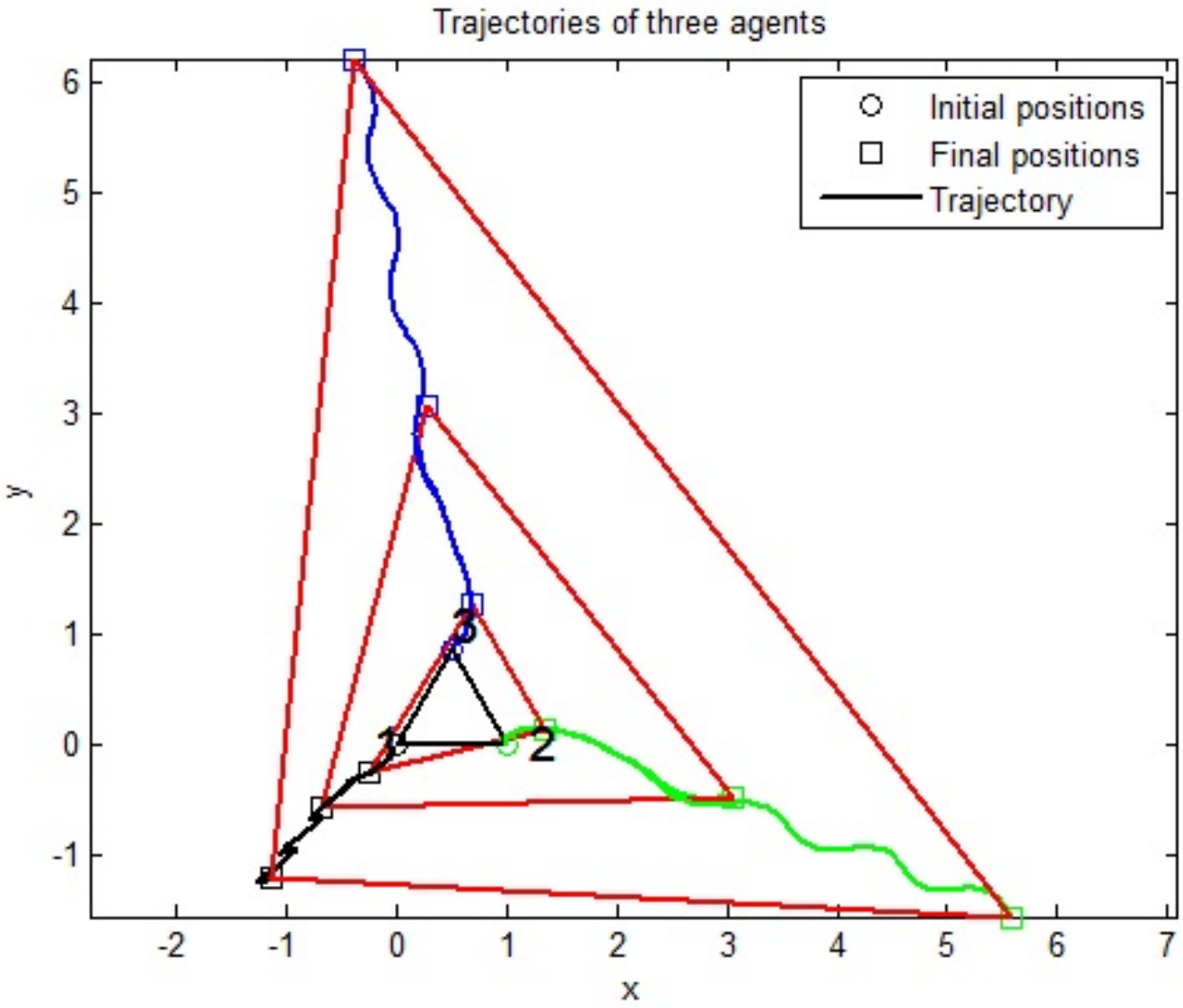
Case 8


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
1.5
1
0.5
0
-0.5
-1
y
-1
-0.5
0
0.5
1
1.5
x

Case 9


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
x
y
-3
-2
-1
0
1
2
3
4
5
6
7
-2
-1
0
1
2
3
4
5

Case 10


Trajectories of three agents
Initial positions
Final positions
Trajectory
1
2
3
x
y
-2
-1
0
1
2
3
4
5
6
7
-1
0
1
2
3
4
5
6

CONTENTS

Motivation

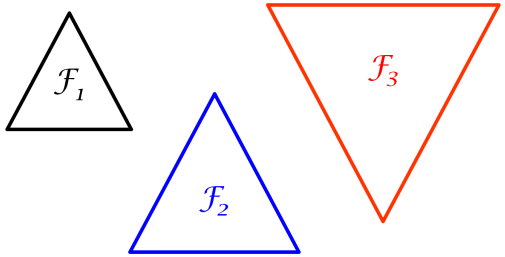


Figure: λ_4 depends on the size of $\mathcal{F}$.

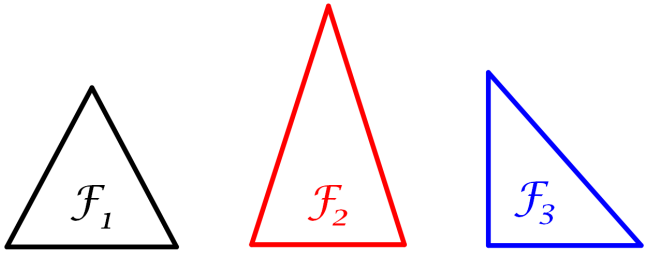


Figure: Which framework is more rigid?

- $\lambda_4(M)$ is usually used as a rigidity index[4]
- $\lambda_4 > 0 \Leftrightarrow \mathcal{F}$ is infinitesimally rigid λ_4 depends *quadratically* on the scale factor ζ.

  $\Rightarrow$ cannot compare rigidity between different frameworks.

[4] **Zelazo2012**.

The new rigidity indices

Consider a framework $\mathcal{F} = (\mathcal{G}, p)$ in the plane,

Definition

The *worst-case rigidity index* of $\mathcal{F}$ is defined as

$$\chi = \frac{\lambda_4}{\sum_{i=1}^{2n} \lambda_i} = \frac{\lambda_4}{\mathrm{tr}(M)}. \tag{8}$$

Definition

The *imbalance index* of the framework $\mathcal{F}$ is defined as

$$\xi = \frac{\lambda_4}{\lambda_{2n}}. \tag{9}$$

PROPERTIES OF THE NEW RIGIDITY INDICES

PROPOSITION

Assume $\mathcal{F}_1 = (\mathcal{G}, p)$ and $\mathcal{F}_2 = (\mathcal{G}, p')$ are two similar frameworks with the worst-case rigidity indices χ_1, χ_2 and the imbalance indices ξ_1, ξ_2. Then $\chi_1 = \chi_2$ and $\xi_1 = \xi_2$.

The new rigidity indices:

- $\chi > 0$ and $\xi > 0 \Leftrightarrow \mathcal{F}$ is infinitesimally rigid
- scale-free
- depend only on the framework's shape.

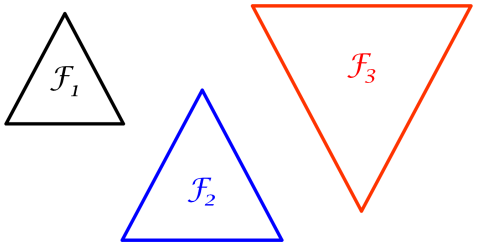
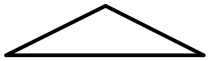
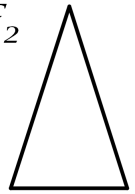
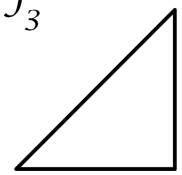
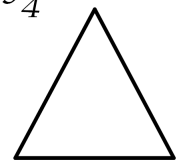
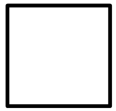
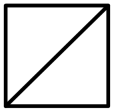


FIGURE: $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ have the same worst-case rigidity index: $\chi_1 = \chi_2 = \chi_3$.

Example 1: Triangular frameworks

	$\mathcal{F}_1$ 	$\mathcal{F}_2$ 	$\mathcal{F}_3$ 	$\mathcal{F}_4$ 
χ	0.0508	0.0853	0.1585	0.25
ξ	0.0726	0.1284	0.2679	0.50

Example 2: Square frameworks

	$\mathcal{F}_1$ 	$\mathcal{F}_2$ 	$\mathcal{F}_3$ 	$\mathcal{F}_4$ 
χ	0	0.0253	0.0637	0.1250
ξ	0	0.0627	0.1459	0.3333

CONTENTS

CONCLUSIONS

Main results:

- Further analysis on the symmetric rigidity matrix M:
  - Physical interpretation of the eigenvectors
  - Further properties of the eigenvalues
- Two scale-free rigidity indices:
  - The worst-case rigidity index χ
  - The imbalance index ξ.

Further studies:

- Find more properties of the rigidity indices
- Relationship between the rigidity matrix and the stiffness matrix
- Extend the results to 3D frameworks.

Q & A †

Thank you!

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References I