Rigidity Theory and Formation Control
A Tutorial

Rigidity Theory for Multi-agent Systems Meets Parallel Robots: Towards the Discovery of Common Models and Methods

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Solutions to coordination problems in multi-robot systems are highly dependent on the sensing and communication mediums available!

Selection criteria depends on mission requirements, cost, environment…

**Sensing**
- GPS
- Relative Position Sensing
- Range Sensing
- Bearing Sensing

**Communication**
- Internet
- Radio
- Sonar
- MANet
Challenges in Multi-Robot Systems

Solutions to coordination problems in multi-robot systems are *highly* dependent on the sensing and communication mediums available!

Selection criteria depends on mission requirements, cost, environment...

Are there **architectural features** of a multi-agent system that are independent of any particular mission or hardware capabilities?
Towards a Multi-Robot Control Architecture

control architecture for a *single* quadrotor
Towards a Multi-Robot Control Architecture

what is the architecture for a *multi-robot* system?

Environment → Environment

Controlled Variables → Controlled Variables

Environment → IL → IL → IL → IL → Inner Loop

Outer Loop → OL → OL → OL → OL → Outer Loop

Mission → Mission

Inner Loop:

Outer Loop:

? → ? → ?
Towards a Multi-Robot Control Architecture

what is the architecture for a multi-robot system?

Connectivity

Ji and Egerstedt, 2007
Dimarogonas and Kyriakopoulos, 2008
Yang et al., 2010
Robuffo Giordano et al., 2013
Towards a Multi-Robot Control Architecture

is connectivity sufficient for higher-level objectives?

**formation control**

**localization**

http://www.commsys.isy.liu.se/en/research
RIGIDITY THEORY

Rigidity Theory provides the correct framework to address many multi-agent mission objectives
The Formation Control Problem

Given a team of robots endowed with the ability to sense relative state information to neighboring robots, design a control for each robot using only *local information* that asymptotically stabilizes the team to a desired formation shape.
The Formation Control Problem

integrator dynamics

\[ \dot{p}_i(t) = u_i(t) \quad i = 1, \ldots, |\mathcal{V}| = N \]

\[ p_i(t) \in \mathbb{R}^d, \quad d = 2, 3 \]

\[ u_i(t) \in \mathbb{R}^d, \quad d = 2, 3 \]

- each agent equipped with onboard sensors
- no communication
- sensing topology is fixed and undirected
- agents can sense relative position of neighboring agents

sensing graph

\[ \mathcal{G} = (\mathcal{V}, \mathcal{E}) \]

\[ (p_i(t) - p_j(t)), \quad \{i, j\} \in \mathcal{E} \]

Geometric Objective

Achieve a desired geometric configuration that can be expressed as a function of relative distances \( \|p_i - p_j\|, \quad \{i, j\} \in \mathcal{G} \).

*adapted from F. Bullo
Lectures on Network Systems
2017
The Formation Control Problem

Geometric Objective expressed as the minimum of some artificial potential function

\[ V_{ij} : D_{ij} \subset \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \]

\[ V_{ij}(\|p_i - p_j\|) \]

\[ f_{ij}(p_i - p_j) = -\frac{\partial}{\partial p_i} V_{ij}(\|p_i - p_j\|) \]

\[ f_{ji}(p_i - p_j) = -\frac{\partial}{\partial p_j} V_{ij}(\|p_i - p_j\|) \]

- potential function for each edge
- assume twice continuously differentiable on its domain
- function of the relative distances

*adapted from F. Bullo
Lectures on Network Systems 2017

3 agents with relative position vectors
artificial potentials as “springs” connecting agents
forces acting on each agent
The Formation Control Problem

Network Potential Function

\[ V(p) = \sum_{\{i,j\} \in \mathcal{E}} V_{ij} \left( \|p_i - p_j\| \right) \]

The Gradient Dynamical System

\[ \dot{p}(t) = -\nabla V(p) \]

\[ \dot{p}_i(t) = -\frac{\partial}{\partial p_i} V(p) \]

\[ = - \sum_{\{i,j\} \in \mathcal{E}} \frac{\partial}{\partial p_i} V_{ij} (\|p_i - p_j\|) \]

\[ = - \sum_{\{i,j\} \in \mathcal{E}} f_{ij}(p_i - p_j), \quad i = \{1, \ldots, N\} \]

*adapted from F. Bullo

_Lectures on Network Systems 2017_
The Formation Control Problem

Distance Potential

\[ V_{ij}(\|p_i - p_j\|) = \frac{1}{4} \left( \|p_i - p_j\|^2 - d_{ij}^2 \right)^2 \]

\[ \dot{p}_i(t) = -\frac{\partial}{\partial p_i} V(p) \]
\[ = -\sum_{\{i,j\} \in \mathcal{E}} \left( \|p_i(t) - p_j(t)\|^2 - d_{ij}^2 \right) (p_i(t) - p_j(t)) \]
The Formation Control Problem

**Theorem**

Consider the potential function with *feasible* target distances

\[ V(p) = \sum_{\{i,j\} \in \mathcal{E}} V_{ij}(\|p_i - p_j\|) = \frac{1}{4} \left( \|p_i - p_j\|^2 - d_{ij}^2 \right)^2 \]

Then the gradient dynamical system

\[ \dot{p}(t) = -\nabla V(p) \]

asymptotically converges to the critical points of the potential function.
The Formation Control Problem

\[
\dot{p}_i(t) = -\frac{\partial}{\partial p_i} V(p) = \sum_{\{i,j\} \in \mathcal{E}} \left( \|p_i(t) - p_j(t)\|^2 - d_{ij}^2 \right) (p_i(t) - p_j(t))
\]
The Formation Control Problem

\[ \dot{p}_i(t) = -\frac{\partial}{\partial p_i} V(p) = \sum_{\{i,j\} \in \mathcal{E}} \left( \|p_i(t) - p_j(t)\|^2 - d^2_{ij} \right) (p_i(t) - p_j(t)) \]
Can the desired formation be maintained using only the available distance measurements?

No!
A minimum number of distance measurements are required to uniquely determine the desired formation!

Graph Rigidity
Bar-and-Joint Frameworks

A framework is a pair \((\mathcal{G}, p)\)

\[
\mathcal{G} = (\mathcal{V}, \mathcal{E}) \\
p : \mathcal{V} \rightarrow \mathbb{R}^2
\]

maps every vertex to a point in the plane

example:

\(\mathcal{F}_1 = (\mathcal{G}, p_1)\)

\(\mathcal{F}_2 = (\mathcal{G}, p_2)\)
Bar-and-Joint Frameworks

bar-and-joint frameworks

\[ G = (V, E) \]

\[ p : V \rightarrow \mathbb{R}^2 \]

maps every vertex to a point in the plane

Two frameworks are equivalent if

\[ (G, p_0) (G, p_1) \]

\[ \forall \{v_i, v_j\} \in E \]

Two frameworks are congruent if

\[ (G, p_0) (G, p_1) \]

\[ \forall v_i, v_j \in V \]
Bar-and-Joint Frameworks

Frameworks are equivalent but not congruent!
Bar-and-Joint Frameworks

Definition

A framework \((\mathcal{G}, p_0)\) is *globally rigid* if every framework that is equivalent to \((\mathcal{G}, p_0)\) is congruent to \((\mathcal{G}, p_0)\).

Frameworks that are both *equivalent* and *congruent* are related by only “trivial” motions:

- translations
- rotations
Bar-and-Joint Frameworks

**Definition**

A framework \((G, p_0)\) is *rigid* if there exists an \(\epsilon > 0\) such that every framework \((G, p_1)\) that is equivalent to \((G, p_0)\) and satisfies \(\|p_0(v) - p_1(v)\| < \epsilon\) for all \(v \in \mathcal{V}\), is congruent to \((G, p_0)\).
Bar-and-Joint Frameworks

\[ p(v_1) = \frac{\epsilon'}{2} \]

\[ p'(v_1) \]

\[ p(v_4) = \frac{\epsilon'}{2} \]

\[ p'(v_4) \]

\[ d = \sqrt{d^2 - \frac{\epsilon'}{2}} \]

\[ p(v_2) = p'(v_2) \]

\[ p(v_3) = p'(v_3) \]
Bar-and-Joint Frameworks

Definition

A *minimally rigid framework* is a rigid framework \((G, p_0)\) such that the removal of any edge in \(G\) results in a non-rigid framework.

![Diagram](rigid-minimally-rigid-not-rigid.png)
Bar-and-Joint Frameworks

parameterizing frameworks by a variable representing “time” allows to consider “motions” of a framework $(G, p, t)$

A trajectory is edge consistent if $\|p(v, t) - p(u, t)\|$ is constant for all $\{v, u\} \in \mathcal{E}$ and all $t$.

edge consistent trajectories generate a family of equivalent frameworks

$$\{p(u) \in \mathbb{R}^2 \mid \|p(u) - p(v)\|^2 = \ell_{uv}^2, \forall \{u, v\} \in \mathcal{E}\}$$

$$\Rightarrow \frac{d}{dt} \|x_u(t) - x_v(t)\| = 0, \forall \{u, v\} \in \mathcal{E}$$

$$\Rightarrow (\dot{x}_u(t) - \dot{x}_v(t))^T (x_u(t) - x_v(t)) = 0 \quad \text{infinitesimal motions}$$
infinitesimal motions define a system of equations...

\[(\xi(v_i) - \xi(v_j))^T (p(v_i) - p(v_j)) = 0\]

**The Rigidity Matrix**

\[ R(p) \in \mathbb{R}^{|E| \times 2|V|} \]

\[
R(p) = \begin{bmatrix}
  p_1^x - p_2^x & p_1^y - p_2^y & p_2^x - p_1^x & p_2^y - p_1^y & 0 & 0 \\
  p_1^x - p_3^x & p_1^y - p_3^y & 0 & 0 & p_3^x - p_1^x & p_3^y - p_1^y \\
  0 & 0 & p_2^x - p_3^x & p_2^y - p_3^y & p_3^x - p_2^x & p_3^y - p_2^y
\end{bmatrix}
\]
Graph Rigidity

another approach...

**Edge ‘Distance’ Function**

\[
f(p) = \frac{1}{2} \begin{bmatrix}
\|p(v_i) - p(v_j)\|^2 \\
\vdots \\
\|p(v_i) - p(v_j)\|^2 \\
\end{bmatrix} \in \mathbb{R}^{|E|}
\]

\(\{v_i, v_j\} \in \mathcal{E}\)

the rigidity matrix is the ‘linear’ term in a Taylor series expansion of the edge function!

\[
f(p + \delta p) = f(p) + \frac{\partial f(p)}{\partial p} \delta p + h.o.t.
\]

**The Rigidity Matrix**

\[
R(p) = \frac{\partial f(p)}{\partial p}
\]
The Rigidity Matrix

\[ R(p) = \frac{\partial f(p)}{\partial p} \]

Lemma 1 (Tay1984) A framework \((G, p)\) is infinitesimally rigid if and only if \(\text{rk}[R] = 2|\mathcal{V}| - 3\)

A framework is \textit{minimally infinitesimally rigid} (MIR) if it is infinitesimally rigid and minimally rigid.

\[ \Rightarrow \text{rk}[R(p)] = 2|\mathcal{V}| - 3 = |\mathcal{E}| \]

MIR frameworks have \textit{full row rank}.
The Formation Control Problem

Distance Potential

\[ V(p) = \sum_{\{i,j\} \in \mathcal{E}} V_{ij}(\|p_i - p_j\|) = \frac{1}{4} \sum_{\{i,j\} \in \mathcal{E}} (\|p_i - p_j\|^2 - d_{ij}^2)^2 \]

\[
\dot{p}_i(t) = -\frac{\partial}{\partial p_i} V(p) \\
= -\sum_{\{i,j\} \in \mathcal{E}} (\|p_i(t) - p_j(t)\|^2 - d_{ij}^2) (p_i(t) - p_j(t))
\]

\[
\dot{p} = -R^T(p)R(p)p - R^T(p)d^2
\]
Formation Control - Stability Analysis

\[ \dot{p} = -R^T(p)R(p)p - R^T(p)d^2 \]

a Lyapunov approach

\[ F(p) = \frac{1}{4} \sigma^T \sigma \]

\[ \frac{d}{dt} F(p) = -\sigma^T R(p)R^T(p)\sigma \leq 0 \]

\[ \frac{d}{dt} F(p) = 0 \iff R^T(p)\sigma = 0 \]

some notations…

\[ e_{ij}(t) = e_k(t) = p_i(t) - p_j(t) \]

\[ k = \{i, j\} \in \mathcal{E} \]

\[ \sigma_k = \|e_k\|^2 - d_k^2 = e_k^T e_k - d_k^2 \]

recall: the potential function defining a gradient dynamical system can serve as a Lyapunov function candidate!
Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

Stability and full row rank

A rigidity matrix with full row rank (i.e. minimally infinitesimally rigid framework) implies

$$\{\sigma \mid R^T(p)\sigma = 0\} = \{0\}$$

$$\frac{d}{dt} F(p) = -\sigma^T R(p)R^T(p)\sigma \leq 0$$

$$\Rightarrow \frac{d}{dt} F(p) = 0 \iff \sigma = 0$$

A positive definite matrix!

Theorem

If the rigidity matrix has full row rank then the distributed distance-based formation control law (exponentially) converges to the specified formation set (locally).
Formation Control

\[ \dot{p} = -R^T(p)R(p)p - R^T(p)d^2 \]

![Graphs and plots showing different formations with varying parameters.](image)
Towards a Multi-Robot Control Architecture

what is the architecture for a *multi-robot* system?

Environment → RIGIDITY → Controlled Variables

Environment → Inner Loop (IL) → Outer Loop (OL) → Mission

**Inner Loop (IL) & Outer Loop (OL)**

Mission
Outlook and Challenges

Sensors

- relative position sensors
- distance sensing
- bearing sensing

Coordinate Frames

- common/global reference frames
- local/body frame

Symmetry

- undirected graphs
- directed graphs

Rigidity Theory

Bearing Rigidity Theory

SE(2) Rigidity Theory

measurement-independent theory?