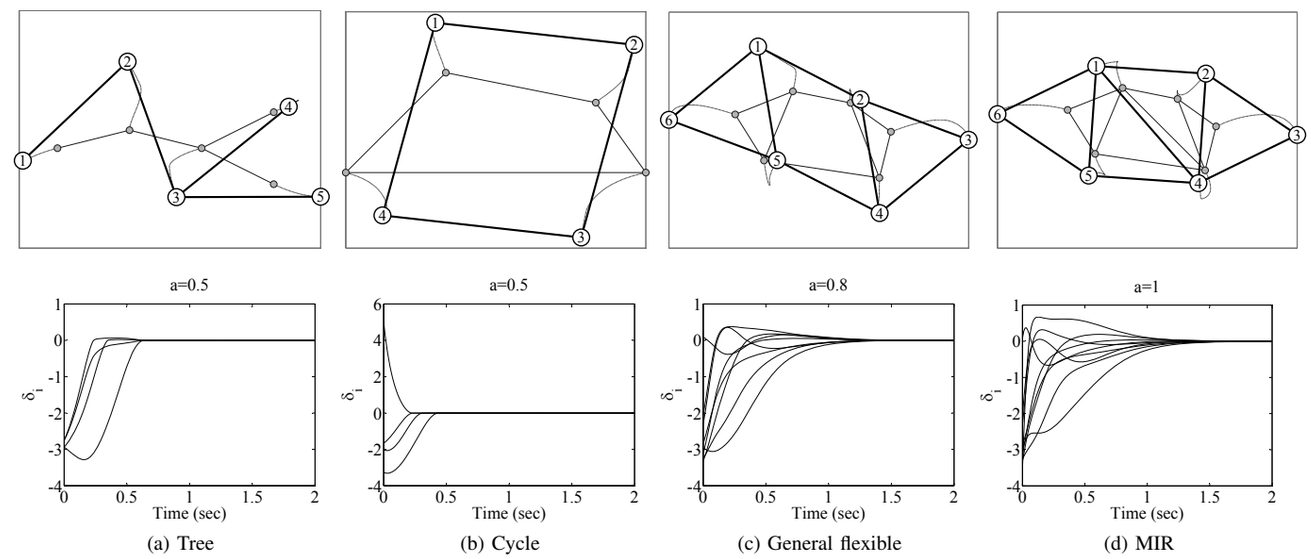
Rigidity Theory for Multi-agent Systems Meets Parallel Robots: Towards the Discovery of Common Models and Methods

Rigidity Theory and Formation Control

A Tutorial



(a) Tree (b) Cycle (c) General flexible (d) MIR

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Challenges in Multi-Robot Systems

Solutions to coordination problems in multi-robot systems are *highly* dependent on the sensing and communication mediums available!

selection criteria depends on mission requirements, cost, environment…

<u>Sensing</u>

- GPS
- Relative Position Sensing
- Range Sensing
- Bearing Sensing

<u>Communication</u>

- Internet
- Radio
- Sonar
- MANet

Challenges in Multi-Robot Systems

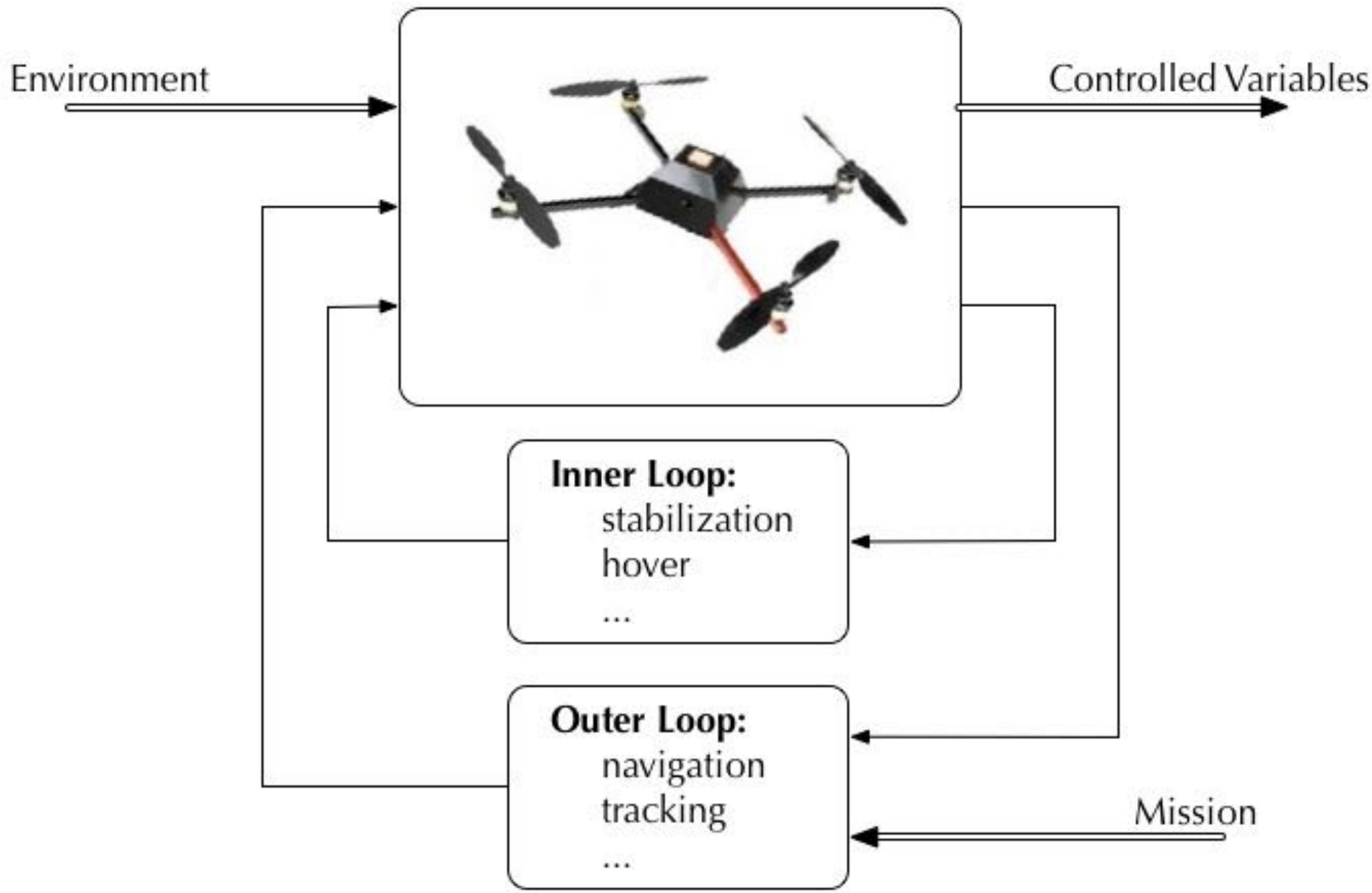
Solutions to coordination problems in multi-robot systems are *highly* dependent on the sensing and communication mediums available!

selection criteria depends on mission requirements, cost, environment…

Are there ***architectural features*** of a multi-agent system that are independent of any particular mission or hardware capabilities?

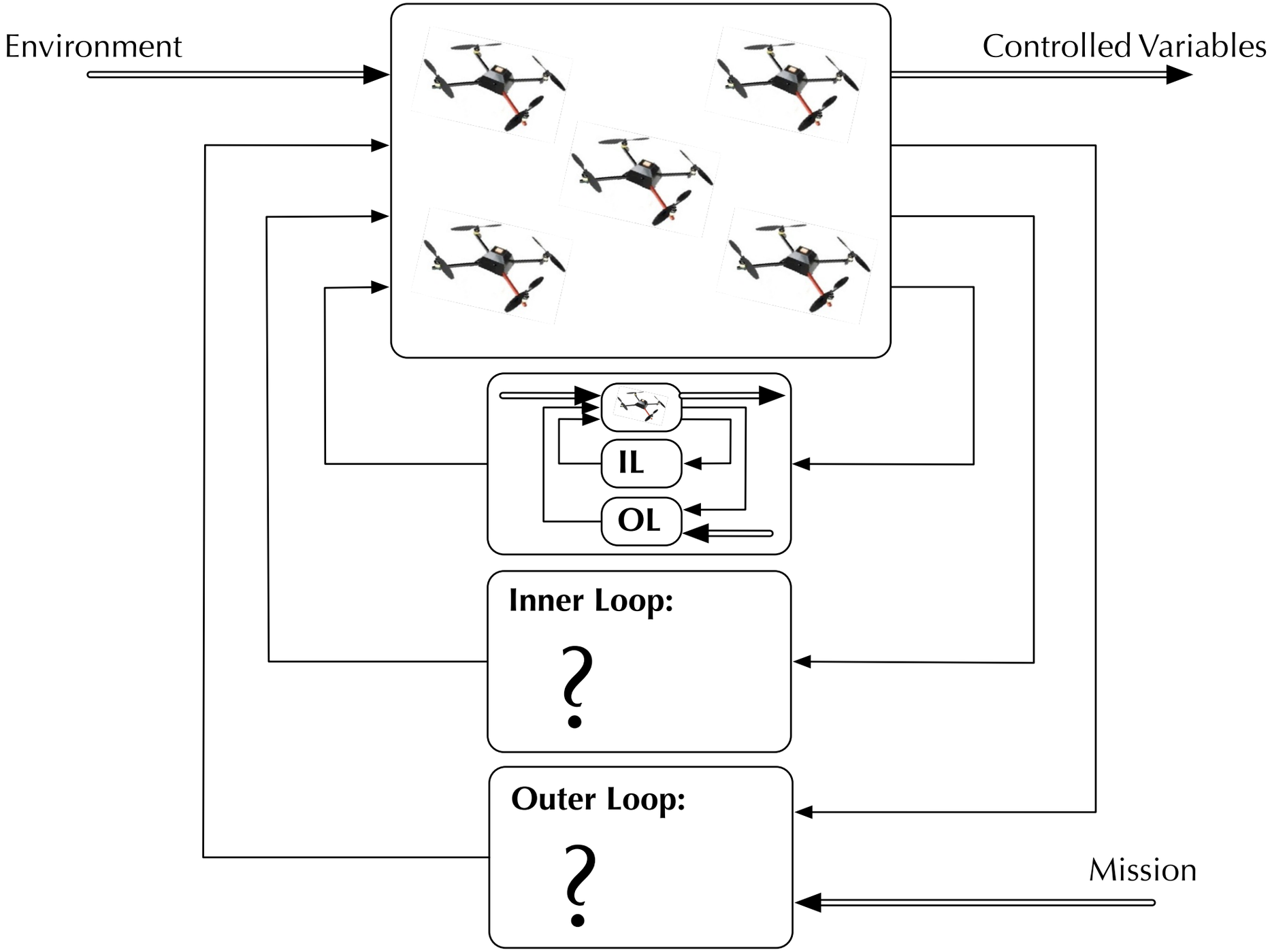
Towards a Multi-Robot Control Architecture

control architecture for a *single* quadrotor



Towards a Multi-Robot Control Architecture

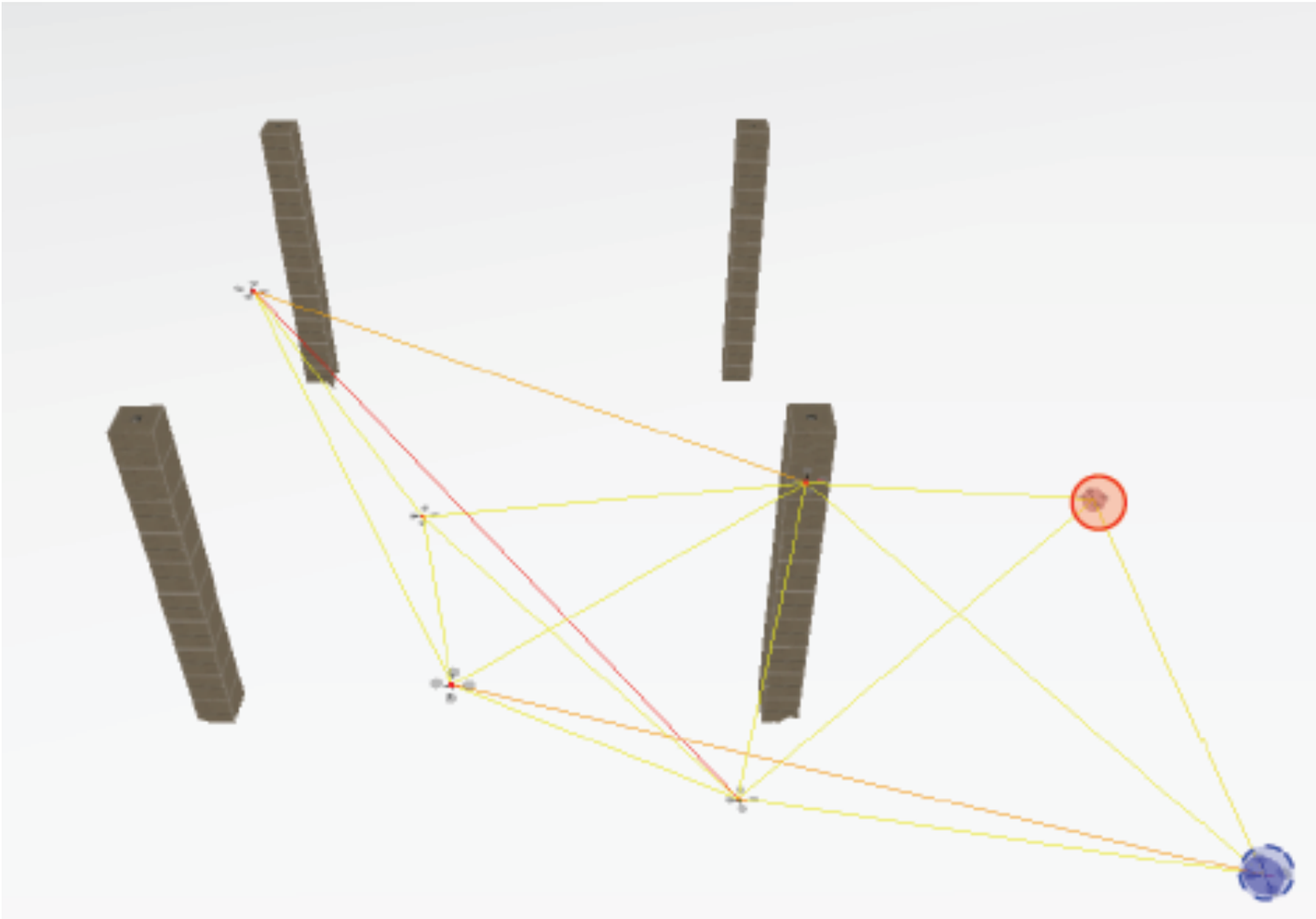
what is the architecture for a *multi-robot* system?



Towards a Multi-Robot Control Architecture

what is the architecture for a *multi-robot* system?

Connectivity

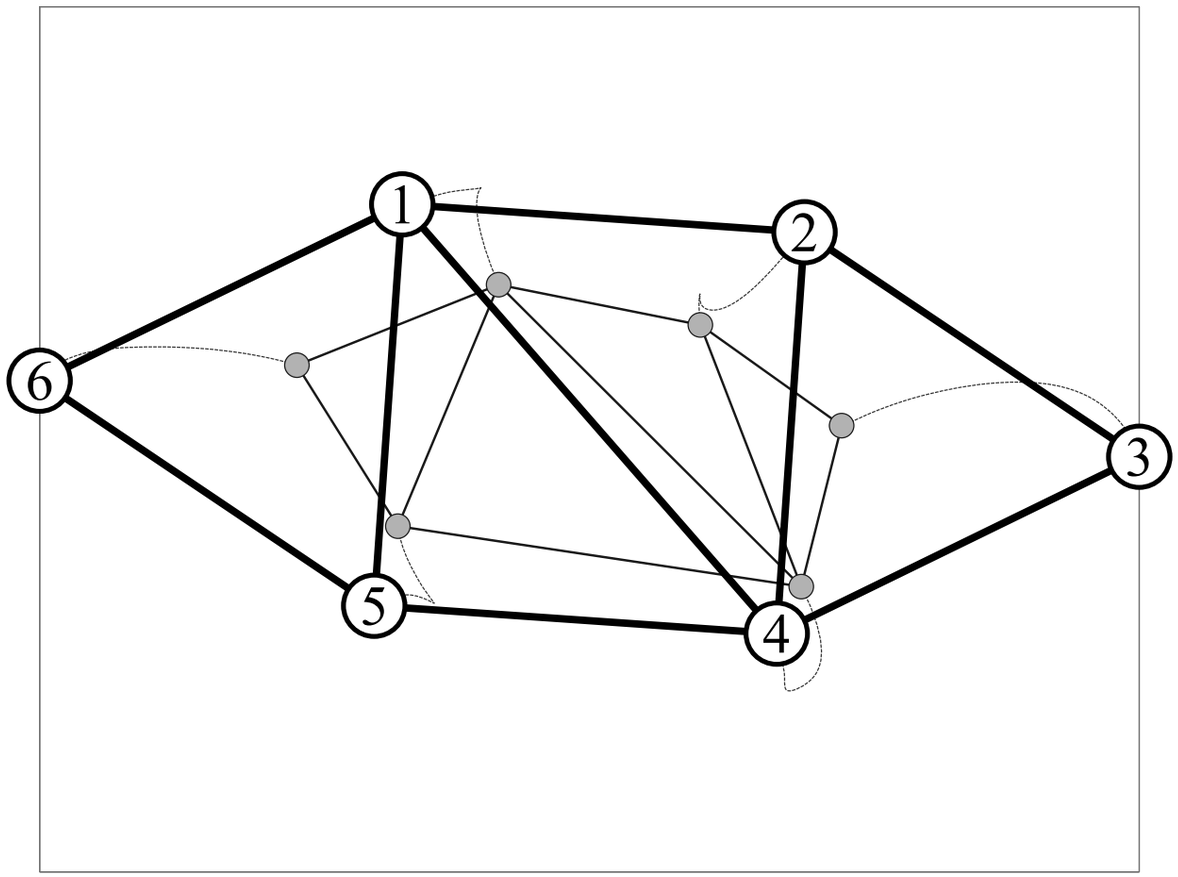


Ji and Egerstedt, 2007
Dimarogonas and Kyriakopoulos, 2008
Yang *et al.*, 2010
Robuffo Giordano *et al.*, 2013

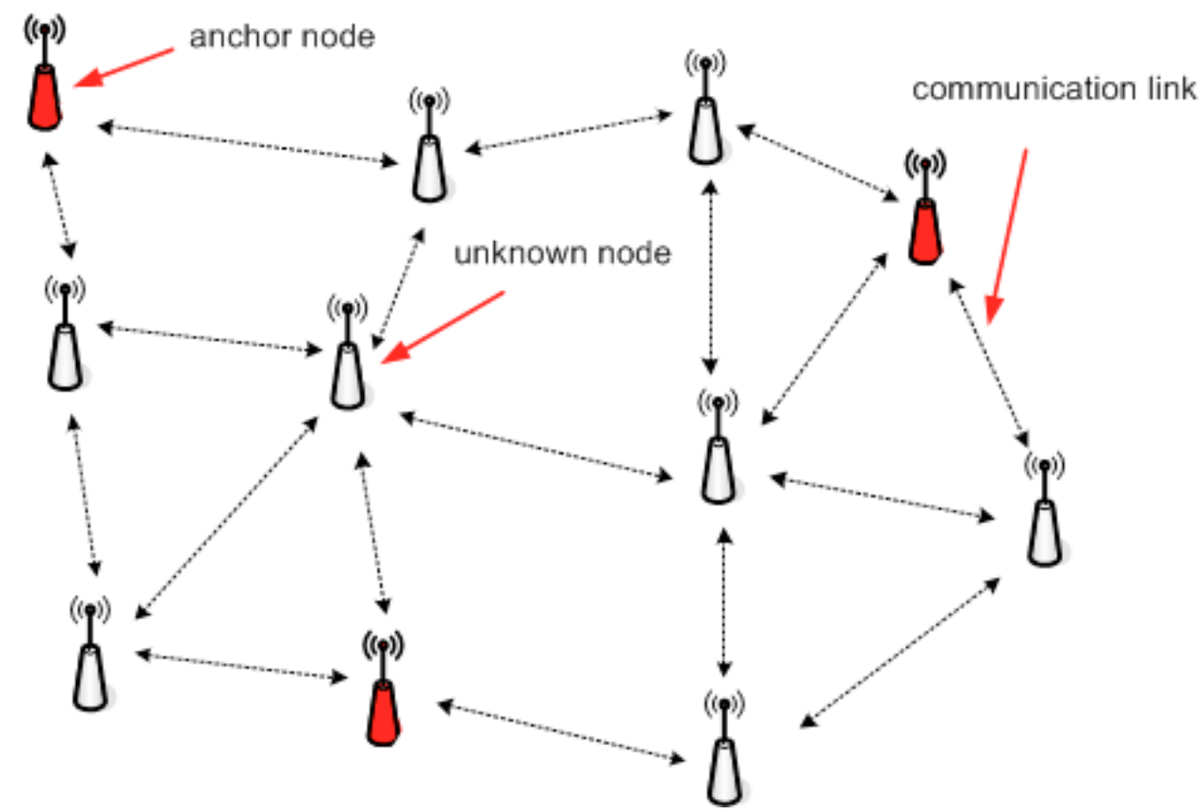
Towards a Multi-Robot Control Architecture

is connectivity sufficient for higher-level objectives?

formation control



localization



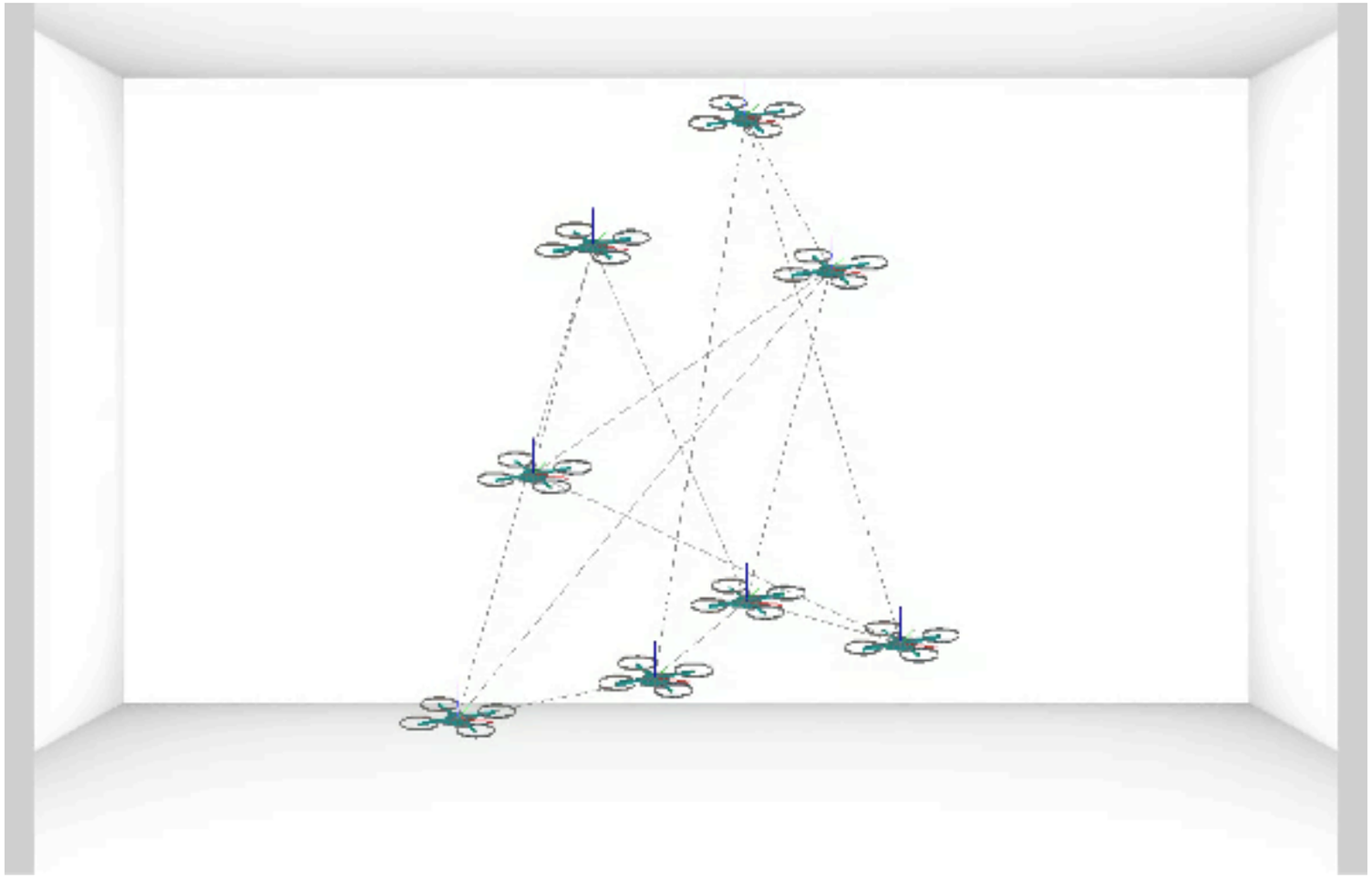
http://www.commsys.isy.liu.se/en/research

RIGIDITY THEORY

Rigidity Theory provides the correct framework to address many multi-agent mission objectives

The Formation Control Problem

Given a team of robots endowed with the ability to sense relative state information to neighboring robots, design a control for each robot using only *local information* that asymptotically stabilizes the team to a desired formation shape.



The Formation Control Problem

integrator dynamics

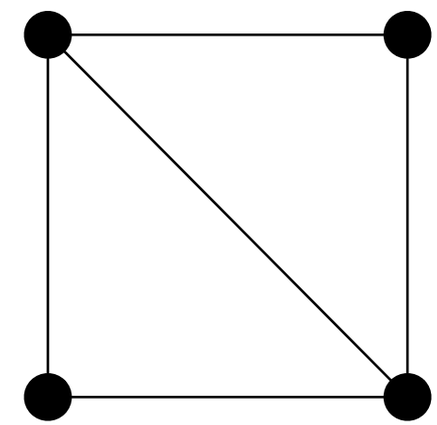
$$\dot{p}_i(t) = u_i(t) \quad i = 1, \ldots, |\mathcal{V}| = N$$

$$p_i(t) \in \mathbb{R}^d, \; d = 2, 3$$

$$u_i(t) \in \mathbb{R}^d, \; d = 2, 3$$

- each agent equipped with onboard sensors
- no communication
- sensing topology is fixed and undirected
- agents can sense *relative position* of neighboring agents

sensing graph



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

$$(p_i(t) - p_j(t)), \; \{i, j\} \in \mathcal{E}$$

Geometric Objective

Achieve a desired geometric configuration that can be expressed as a function of relative distances $\|p_i - p_j\|, \; \{i, j\} \in \mathcal{G}$.

*adapted from F. Bullo
Lectures on Network Systems
2017

The Formation Control Problem

Geometric Objective expressed as the *minimum* of some artificial potential function

$$V_{ij} \; : \; D_{ij} \subset \mathbb{R} \; \rightarrow \; \mathbb{R}_{\geq 0}$$

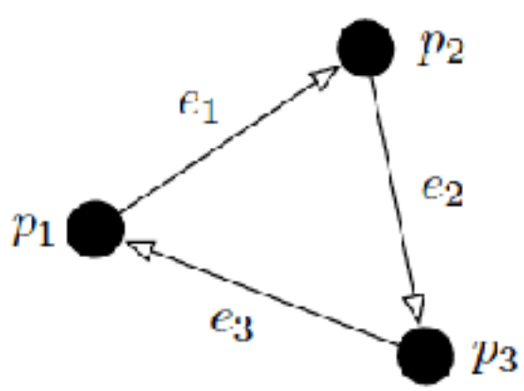
$$V_{ij}(\|p_i - p_j\|)$$

- potential function for each edge
- assume twice continuously differentiable on its domain
- function of the relative distances

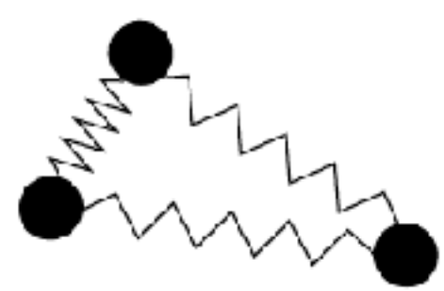
$$f_{ij}(p_i - p_j) = -\frac{\partial}{\partial p_i} V_{ij}(\|p_i - p_j\|)$$

$$f_{ji}(p_i - p_j) = -\frac{\partial}{\partial p_j} V_{ij}(\|p_i - p_j\|)$$

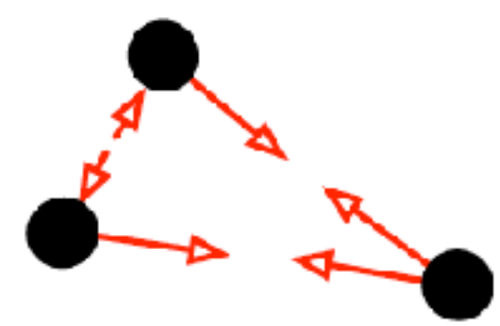
- the "spring force"



3 agents with relative position vectors



artificial potentials as "springs" connecting agents



forces acting on each agent

*adapted from F. Bullo *Lectures on Network Systems* 2017

The Formation Control Problem

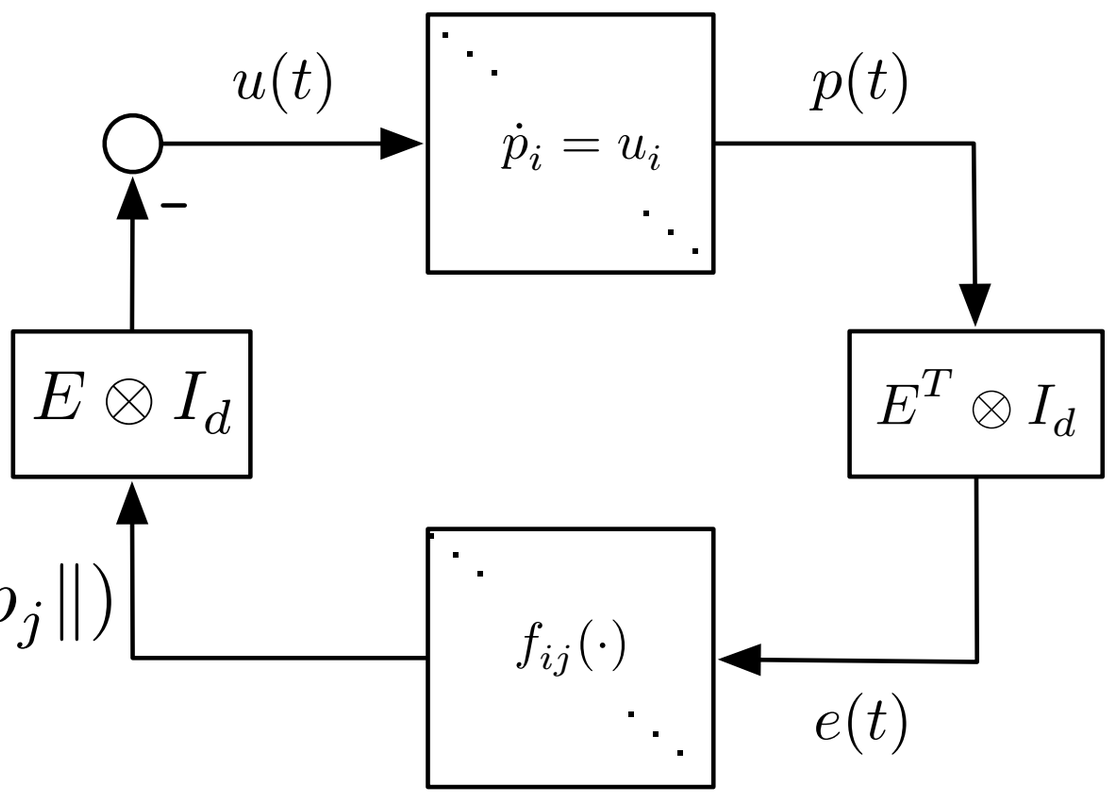
Network Potential Function

$$V(p) = \sum_{\{i,j\}\in\mathcal{E}} V_{ij}(\|p_i - p_j\|)$$

The Gradient Dynamical System

$$\dot{p}(t) = -\nabla V(p)$$

$$\begin{aligned}\dot{p}_i(t) &= -\frac{\partial}{\partial p_i} V(p) \\ &= -\sum_{\{i,j\}\in\mathcal{E}} \frac{\partial}{\partial p_i} V_{ij}(\|p_i - p_j\|) \\ &= -\sum_{\{i,j\}\in\mathcal{E}} f_{ij}(p_i - p_j),\ i = \{1,\ldots,N\}\end{aligned}$$

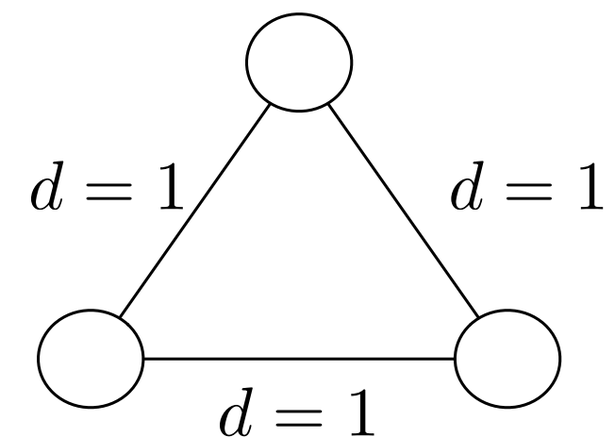


*adapted from F. Bullo
Lectures on Network Systems
2017

The Formation Control Problem

Distance Potential

$$V_{ij}(\|p_i - p_j\|) = \frac{1}{4}\left(\|p_i - p_j\|^2 - d_{ij}^2\right)^2$$



$$\begin{aligned}\dot{p}_i(t) &= -\frac{\partial}{\partial p_i} V(p) \\ &= -\sum_{\{i,j\}\in\mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - d_{ij}^2\right)(p_i(t) - p_j(t))\end{aligned}$$

The Formation Control Problem

Theorem

Consider the potential function with *feasible* target distances

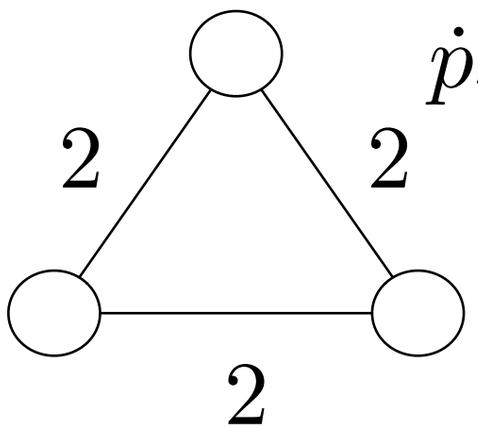
$$V(p) = \sum_{\{i,j\}\in\mathcal{E}} V_{ij}(\|p_i - p_j\|) = \frac{1}{4}\left(\|p_i - p_j\|^2 - d_{ij}^2\right)^2$$

Then the gradient dynamical system

$$\dot{p}(t) = -\nabla V(p)$$

asymptotically converges to the critical points of the potential function.

The Formation Control Problem



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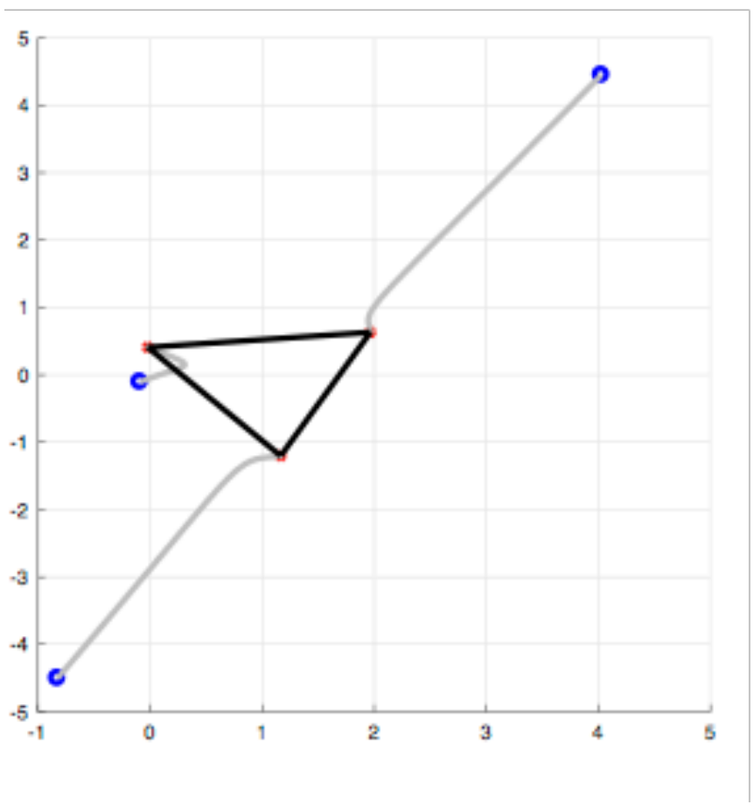
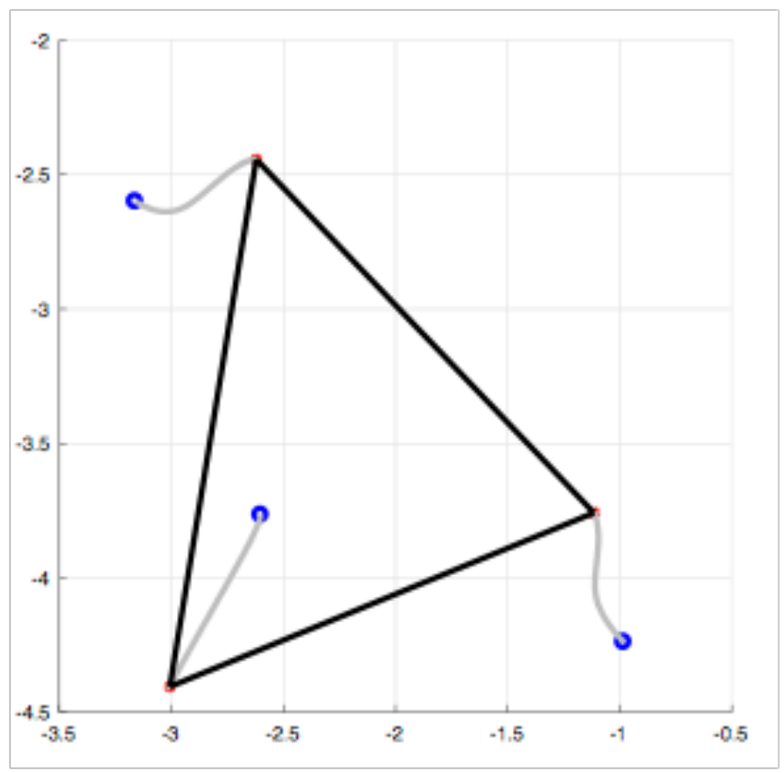
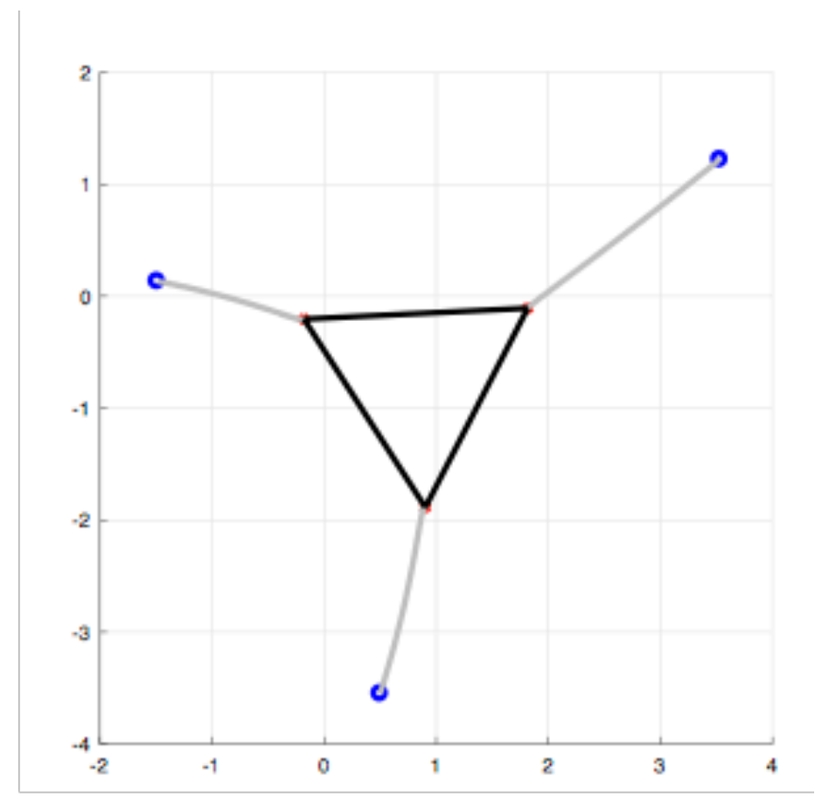
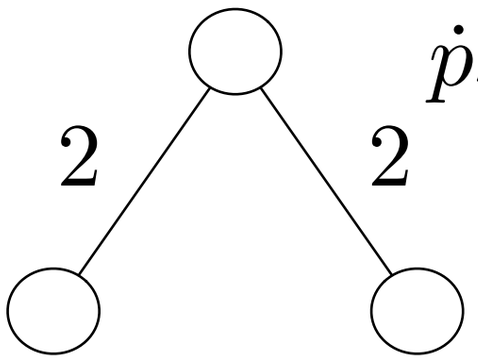


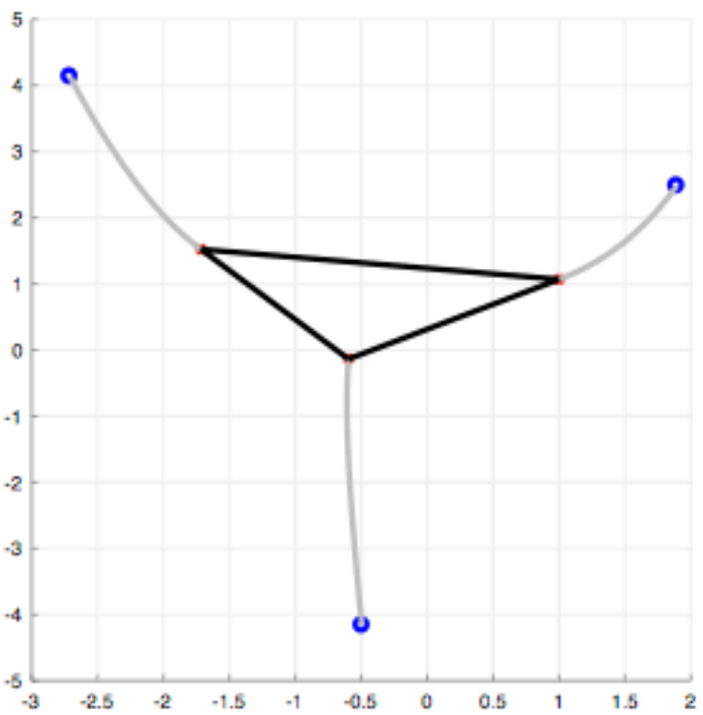


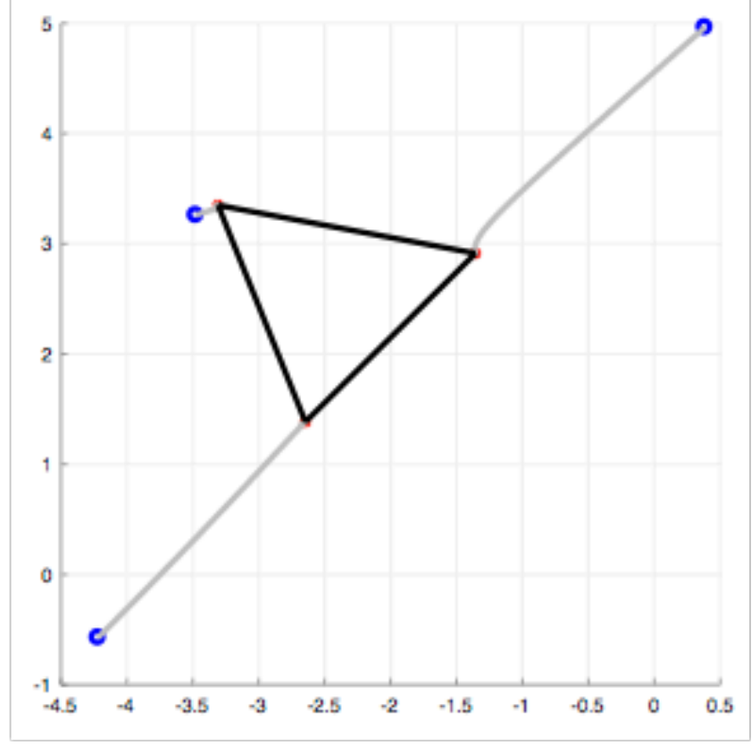


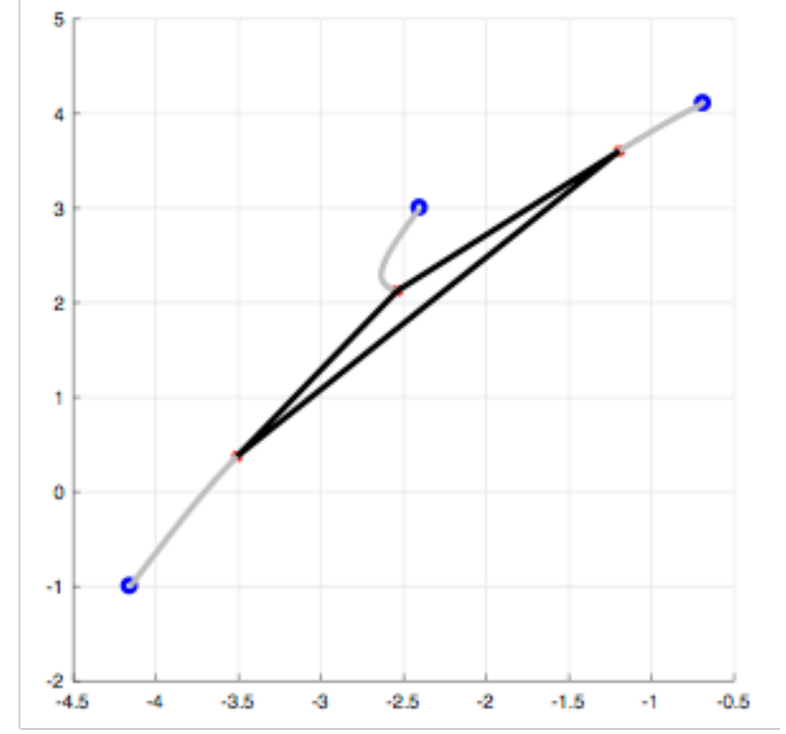
The Formation Control Problem



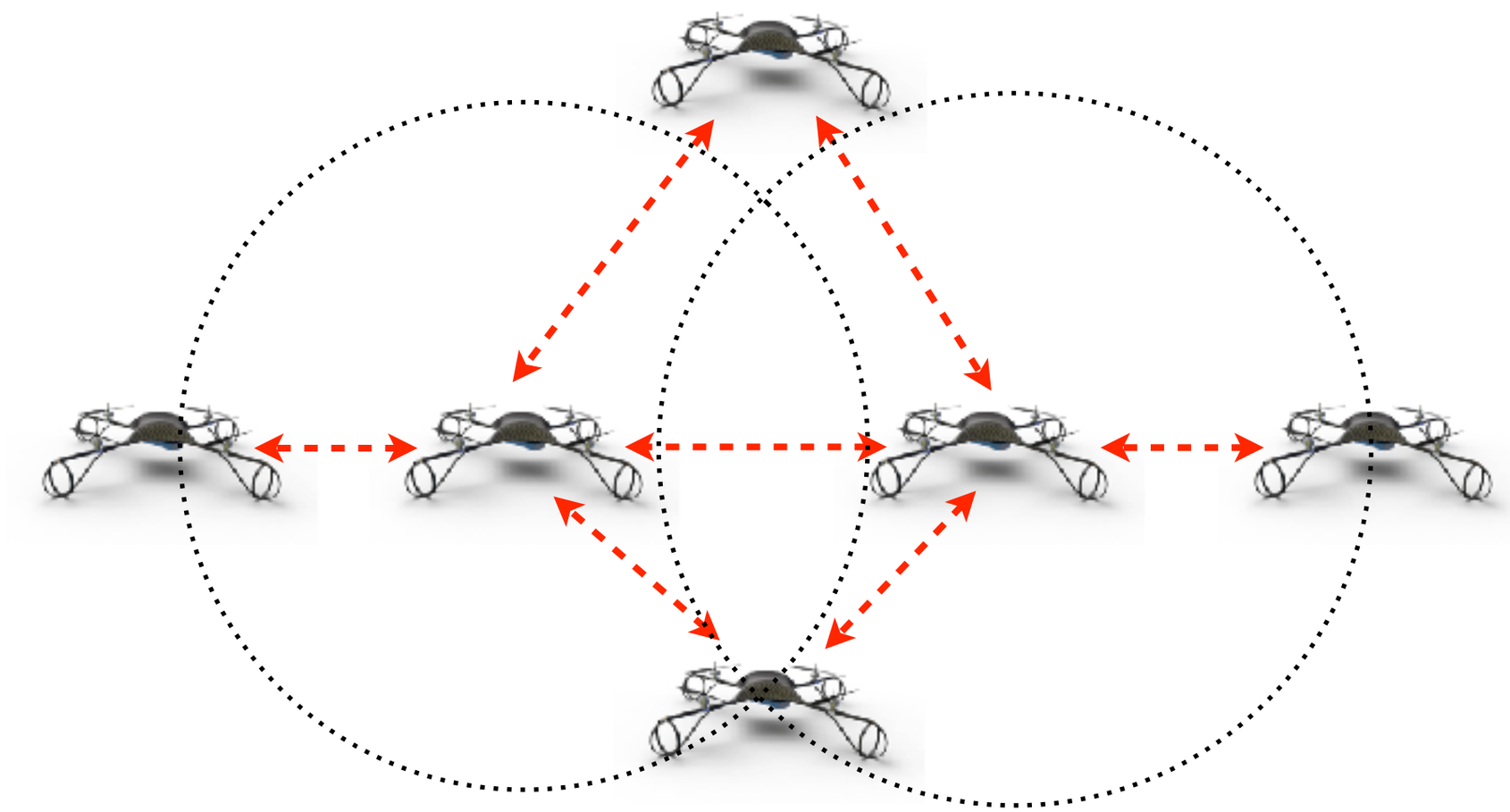
$$\begin{aligned}\dot{p}_i(t) &= -\frac{\partial}{\partial p_i} V(p) \\ &= \sum_{\{i,j\}\in\mathcal{E}} \left(\|p_i(t)-p_j(t)\|^2 - d_{ij}^2\right)\left(p_i(t)-p_j(t)\right)\end{aligned}$$







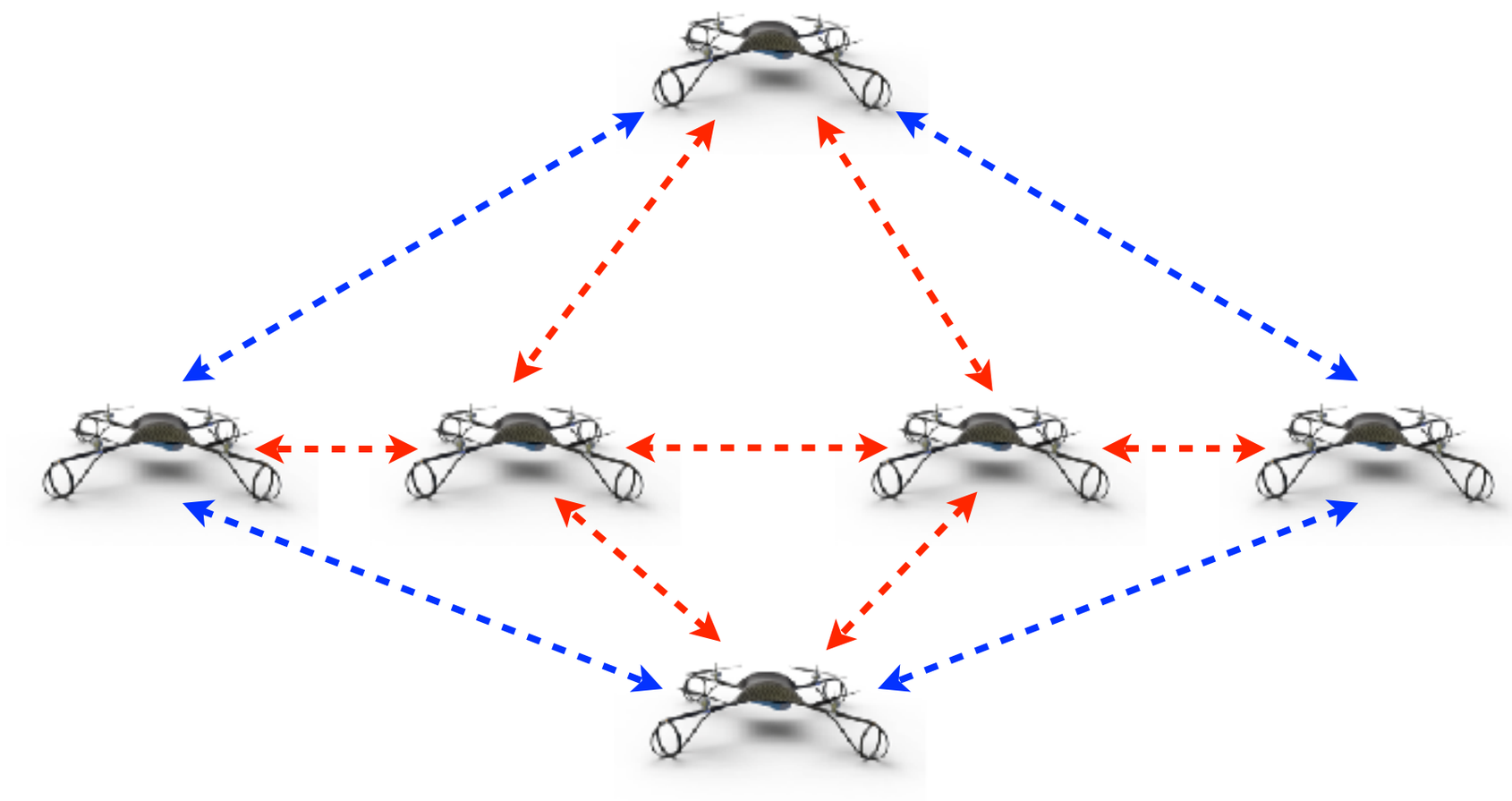
Architectural Requirements



Can the desired formation be maintained using only the available distance measurements?

No!

Architectural Requirements



A *minimum* number of distance measurements are required to *uniquely* determine the desired formation!

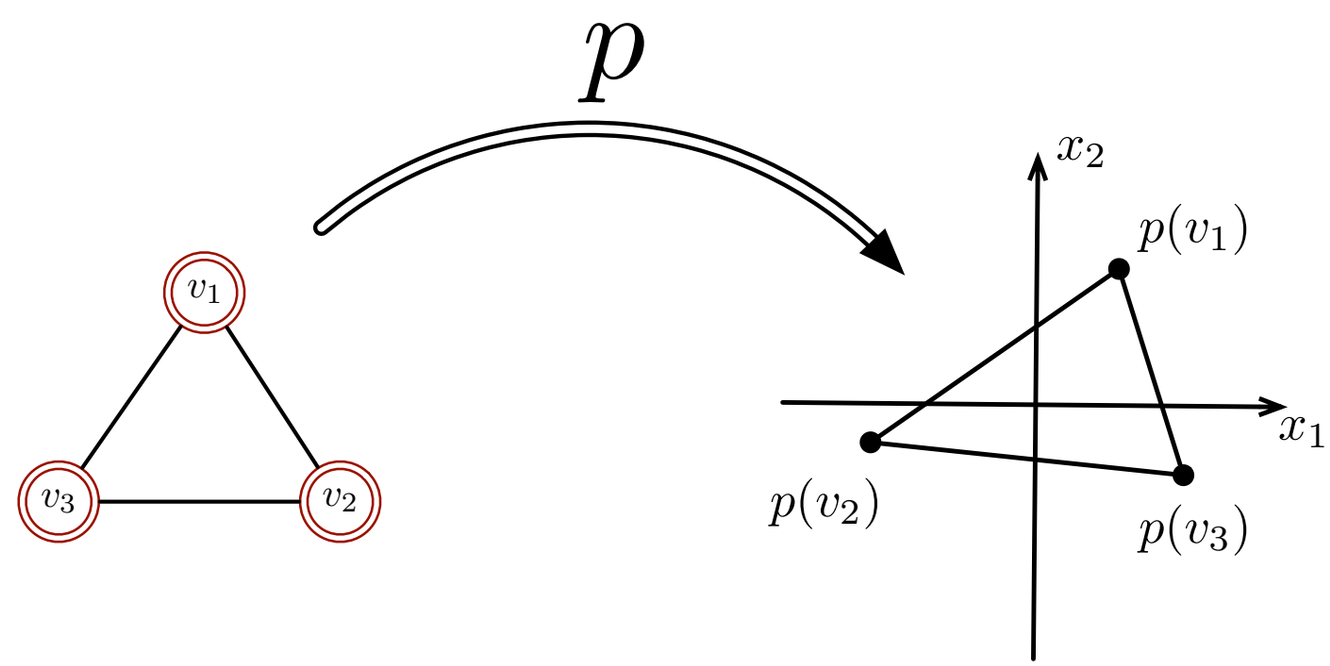
Graph Rigidity

Bar-and-Joint Frameworks

A *framework* is a pair $(\mathcal{G}, p)$

$$\begin{cases} \mathcal{G} = (\mathcal{V}, \mathcal{E}) \\ p \,:\, \mathcal{V} \to \mathbb{R}^2 \end{cases}$$

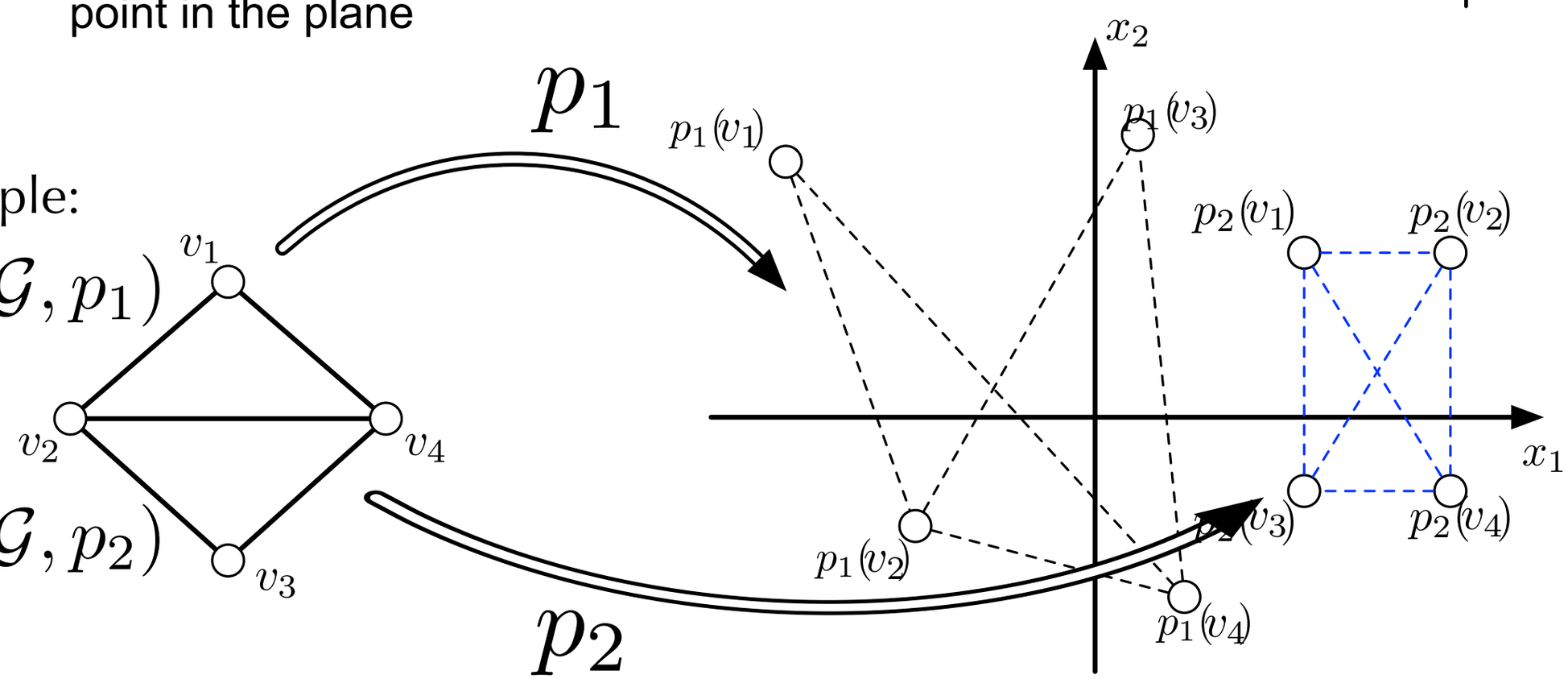
maps every vertex to a point in the plane



example:

$\mathcal{F}_1 = (\mathcal{G}, p_1)$

$\mathcal{F}_2 = (\mathcal{G}, p_2)$



Bar-and-Joint Frameworks

bar-and-joint frameworks

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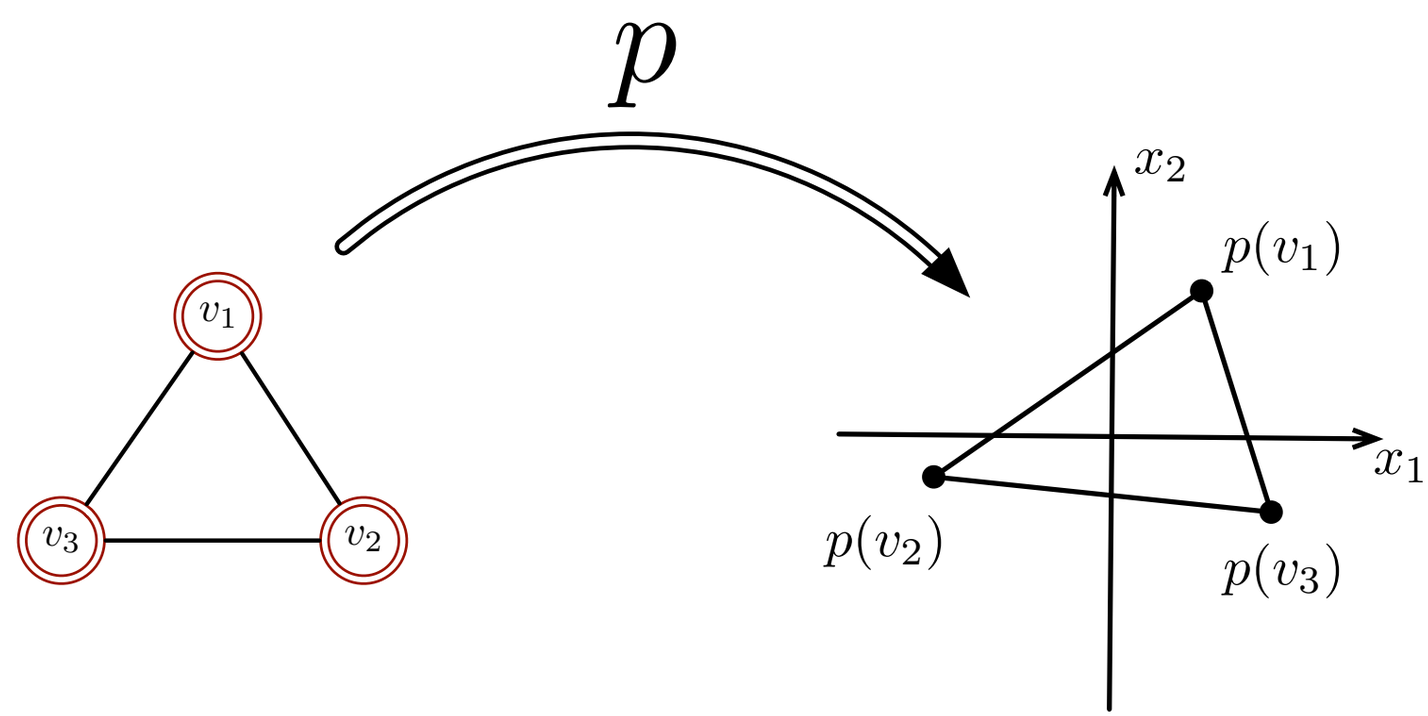


Two frameworks are *equivalent if*

$$(\mathcal{G}, p_0) \quad (\mathcal{G}, p_1)$$

$$\|p_0(v_i) - p_0(v_j)\| = \|p_1(v_i) - p_1(v_j)\|$$

$$\boxed{\forall \{v_i, v_j\} \in \mathcal{E}}$$

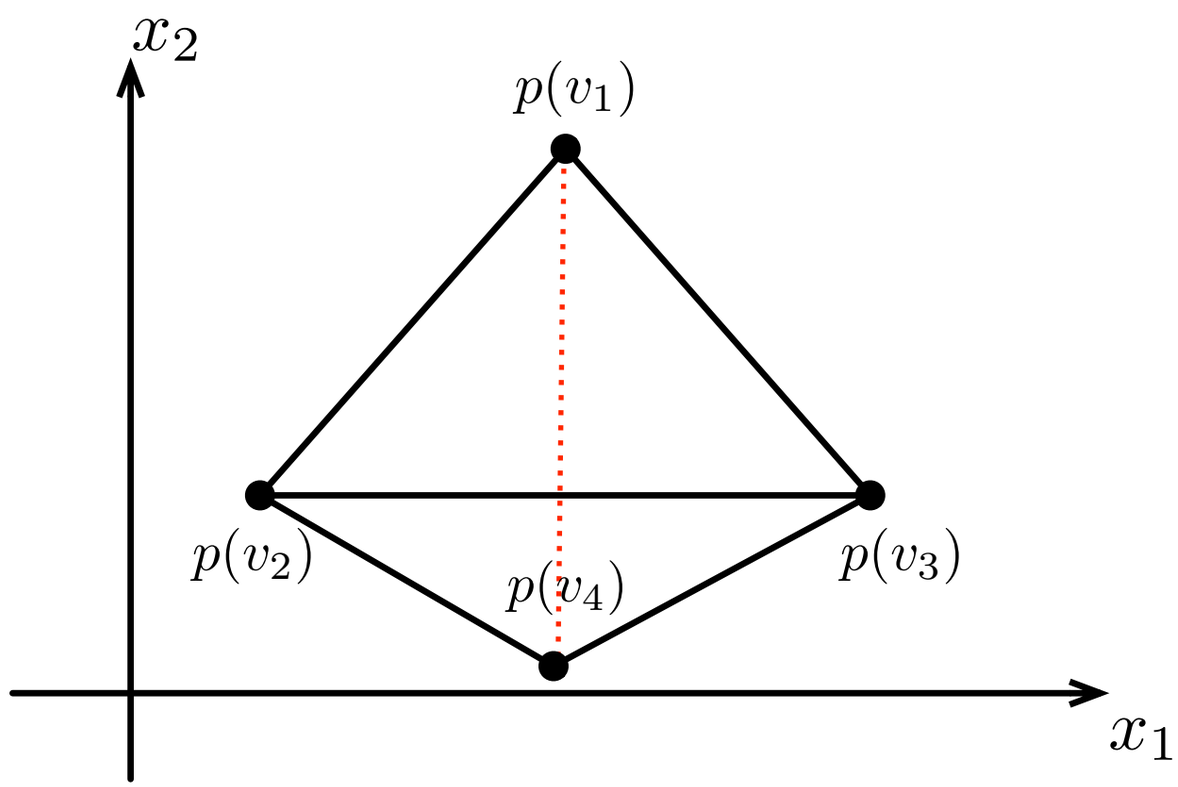
Two frameworks are *congruent if*

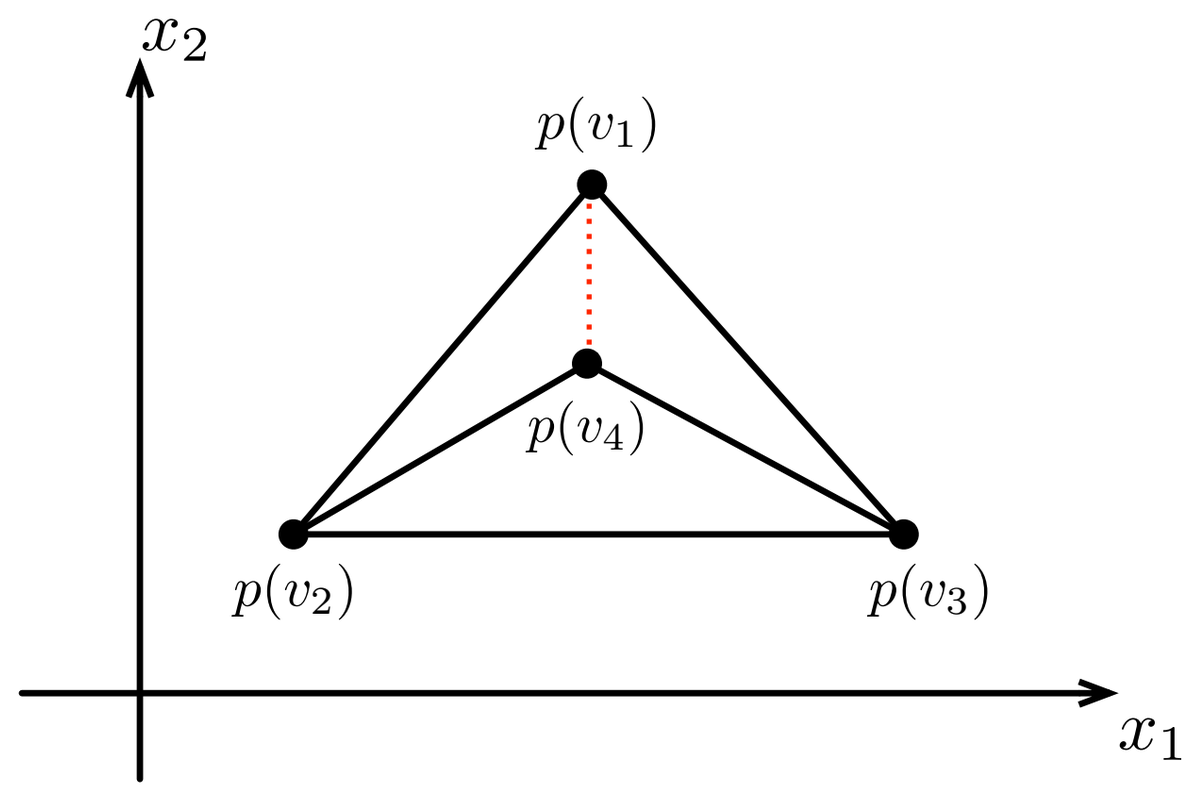
$$(\mathcal{G}, p_0) \quad (\mathcal{G}, p_1)$$

$$\|p_0(v_i) - p_0(v_j)\| = \|p_1(v_i) - p_1(v_j)\|$$

$$\boxed{\forall\, v_i, v_j \in \mathcal{V}}$$

Bar-and-Joint Frameworks





Frameworks are equivalent but not congruent!

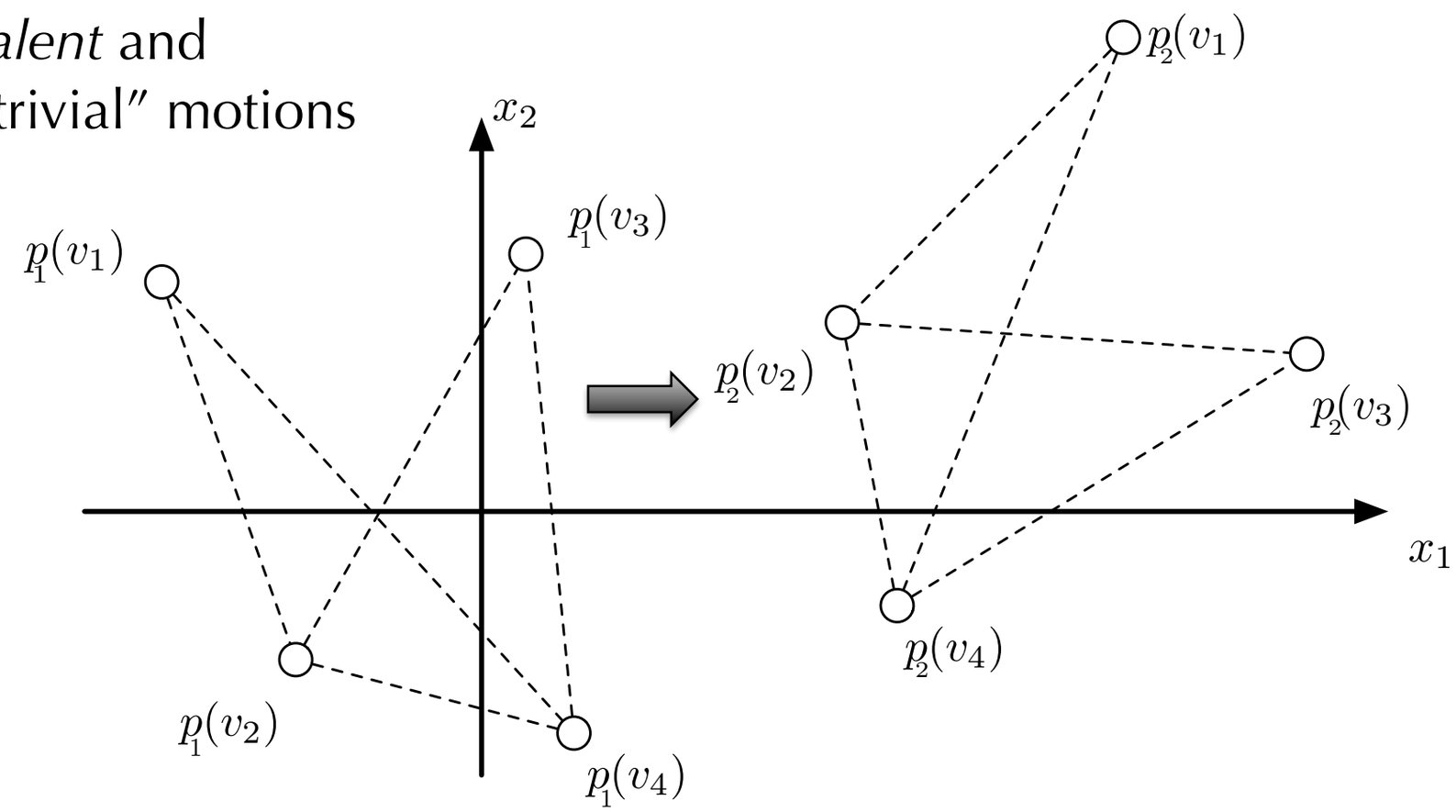
Bar-and-Joint Frameworks

Definition

A framework $(\mathcal{G}, p_0)$ is *globally rigid* if every framework that is equivalent to $(\mathcal{G}, p_0)$ is congruent to $(\mathcal{G}, p_0)$.

frameworks that are both *equivalent* and *congruent* are related by only "trivial" motions

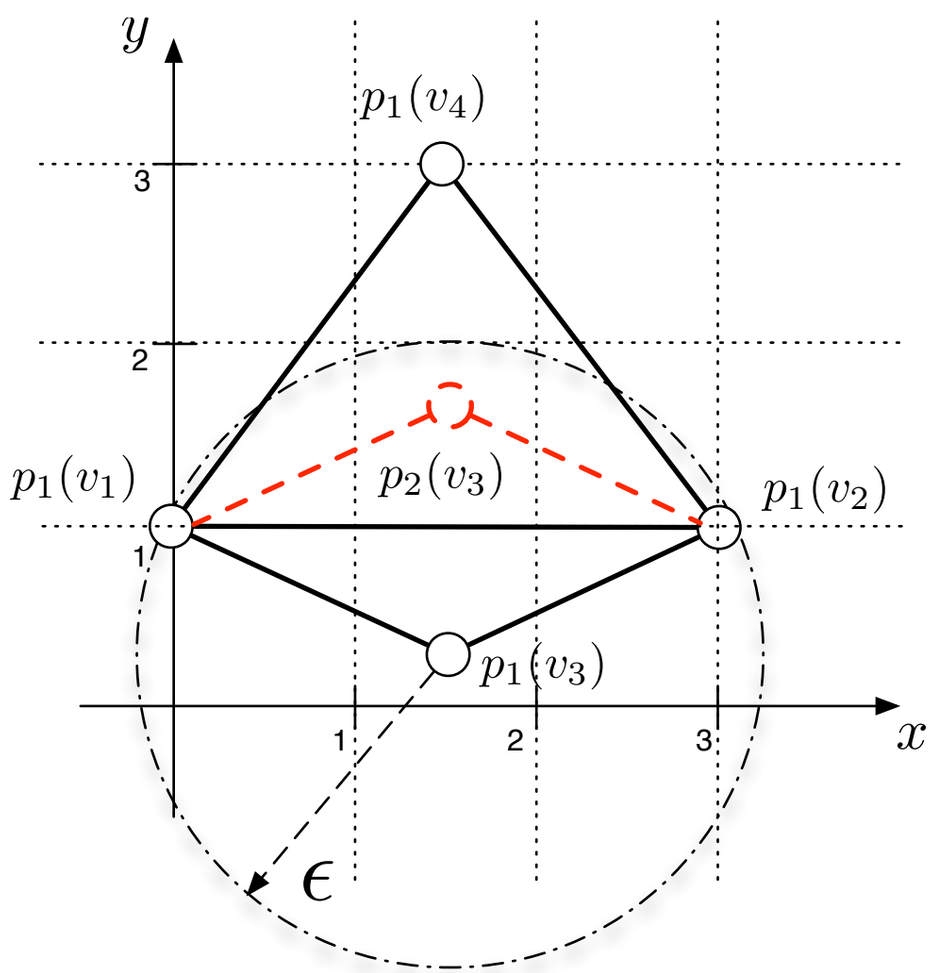
- translations
- rotations

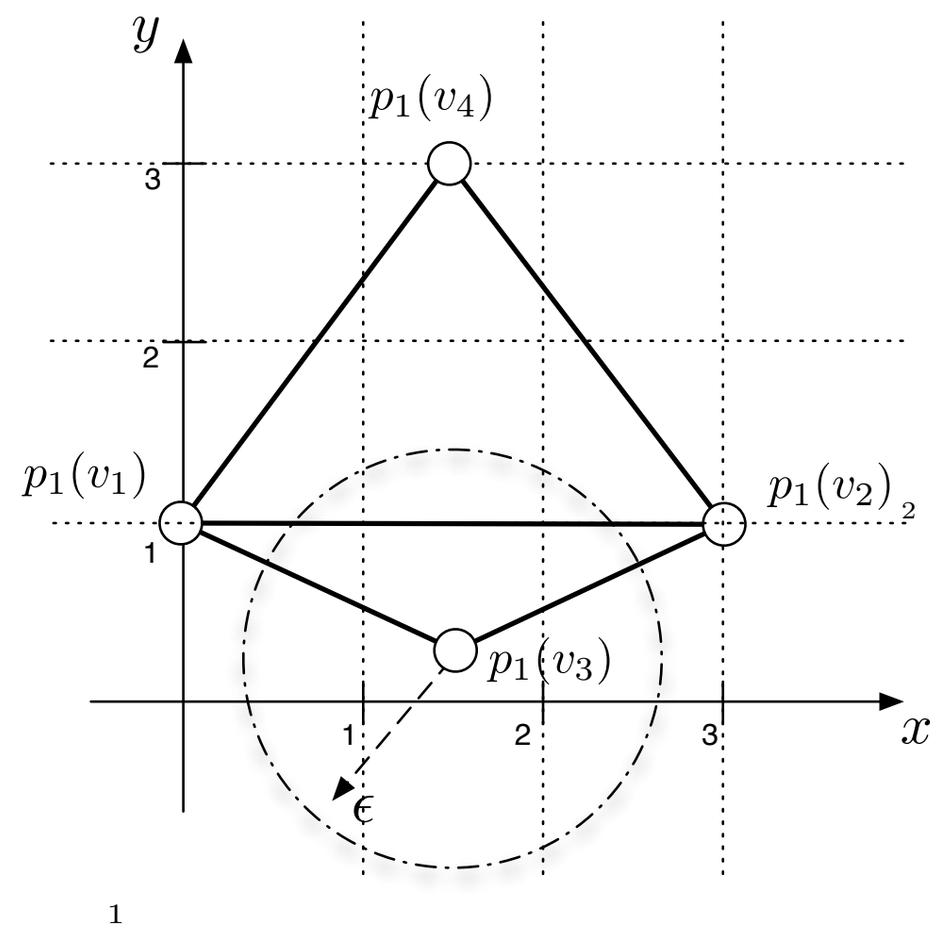


Bar-and-Joint Frameworks

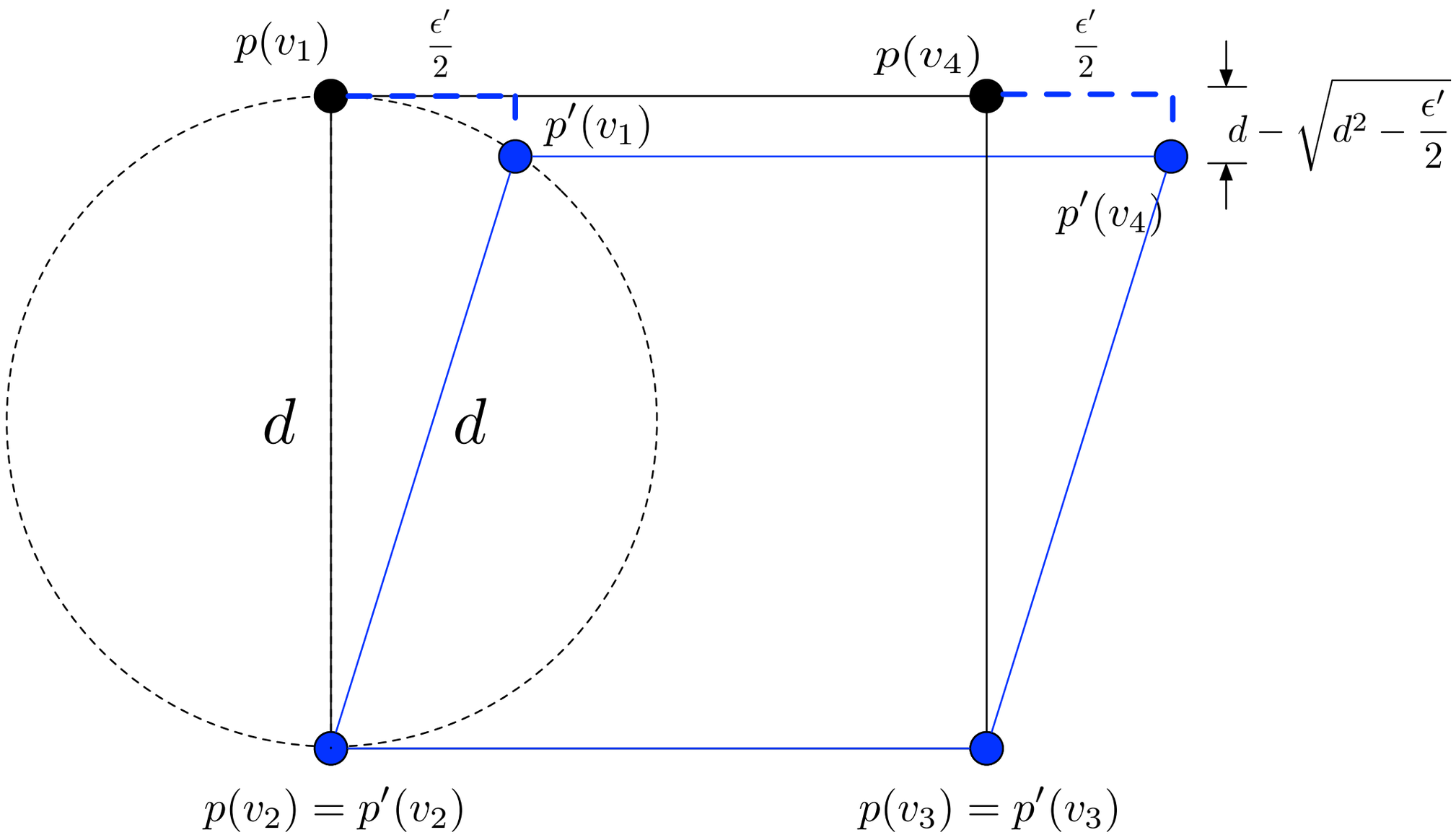
Definition

A framework $(\mathcal{G}, p_0)$ is *rigid* if there exists an $\epsilon > 0$ such that every framework $(\mathcal{G}, p_1)$ that is equivalent to $(\mathcal{G}, p_0)$ and satisfies $\|p_0(v) - p_1(v)\| < \epsilon$ for all $v \in \mathcal{V}$, is congruent to $(\mathcal{G}, p_0)$.





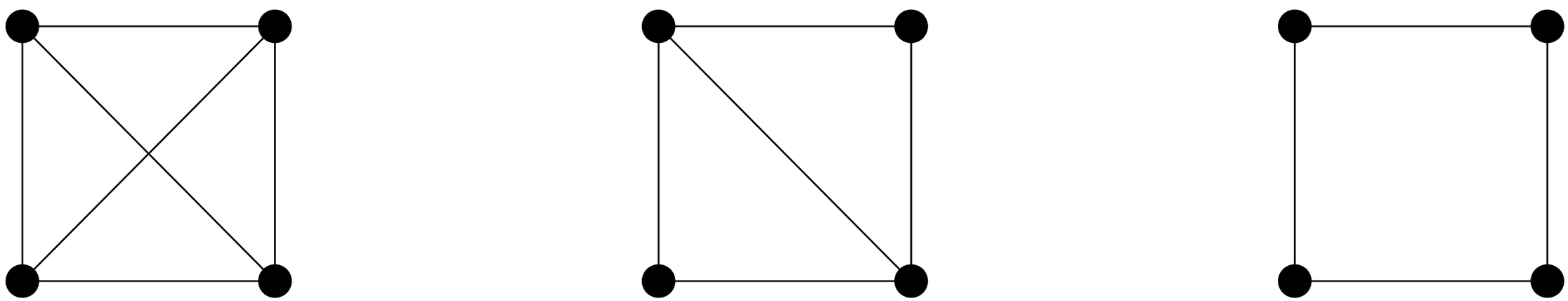
Bar-and-Joint Frameworks



Bar-and-Joint Frameworks

Definition

A *minimally rigid framework* is a rigid framework $(\mathcal{G}, p_0)$ such that the removal of any edge in $\mathcal{G}$ results in a non-rigid framework.



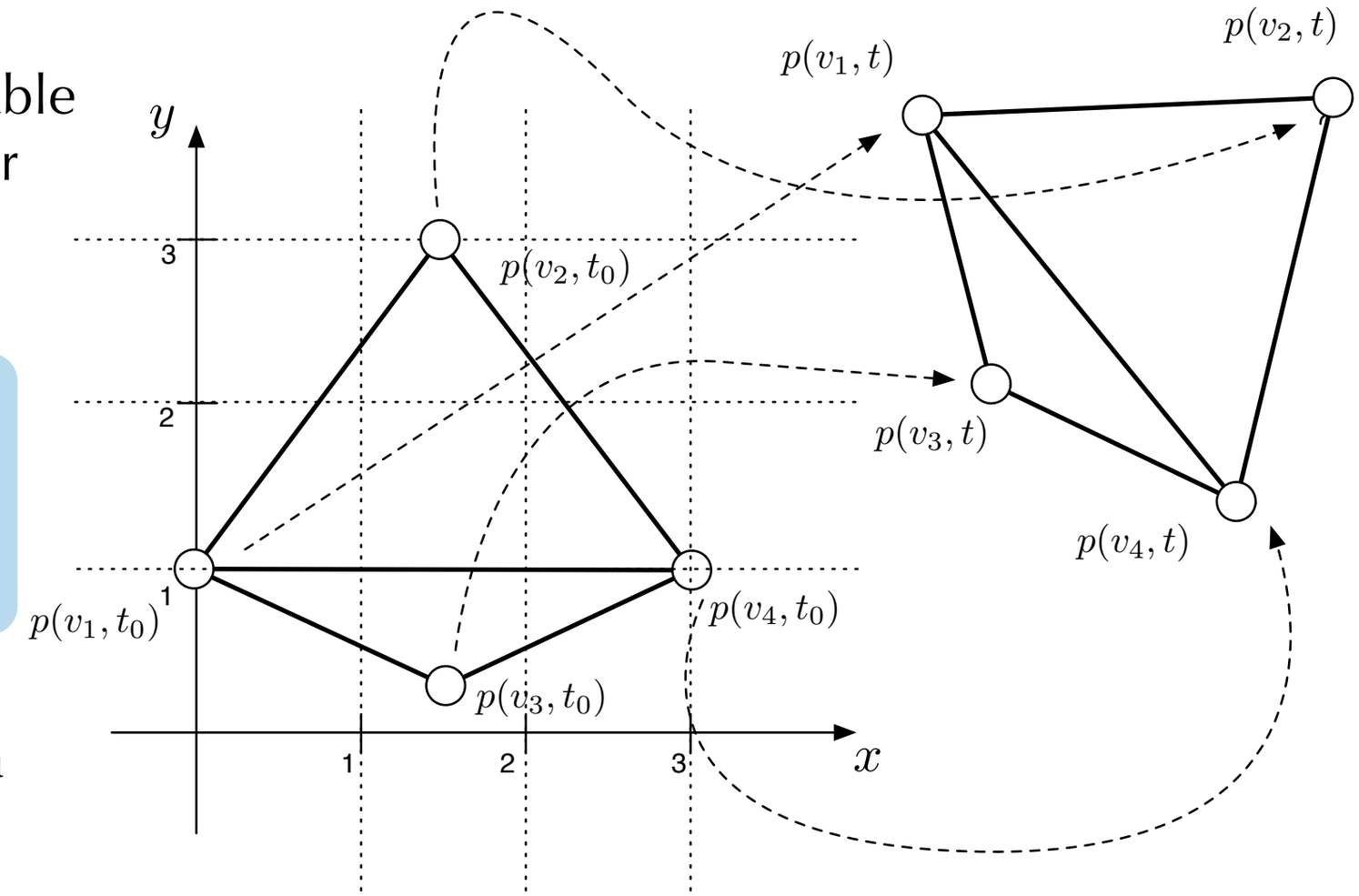
rigid minimally rigid not rigid

Bar-and-Joint Frameworks

parameterizing frameworks by a variable representing "time" allows to consider "motions" of a framework

$(\mathcal{G}, p, t)$

A trajectory is *edge consistent* if $\|p(v,t) - p(u,t)\|$ is constant for all $\{v,u\} \in \mathcal{E}$ and all t.



edge consistent trajectories generate a family of equivalent frameworks

$$\{p(u) \in \mathbb{R}^2 \mid \|p(u) - p(v)\|_2^2 = \ell_{uv}^2, \forall \{u,v\} \in \mathcal{E}\}$$

$$\Rightarrow \frac{d}{dt}\|x_u(t) - x_v(t)\| = 0, \forall \{u,v\} \in \mathcal{E}$$

$$\Rightarrow (\dot{x}_u(t) - \dot{x}_v(t))^T (x_u(t) - x_v(t)) = 0$$

infinitesimal motions

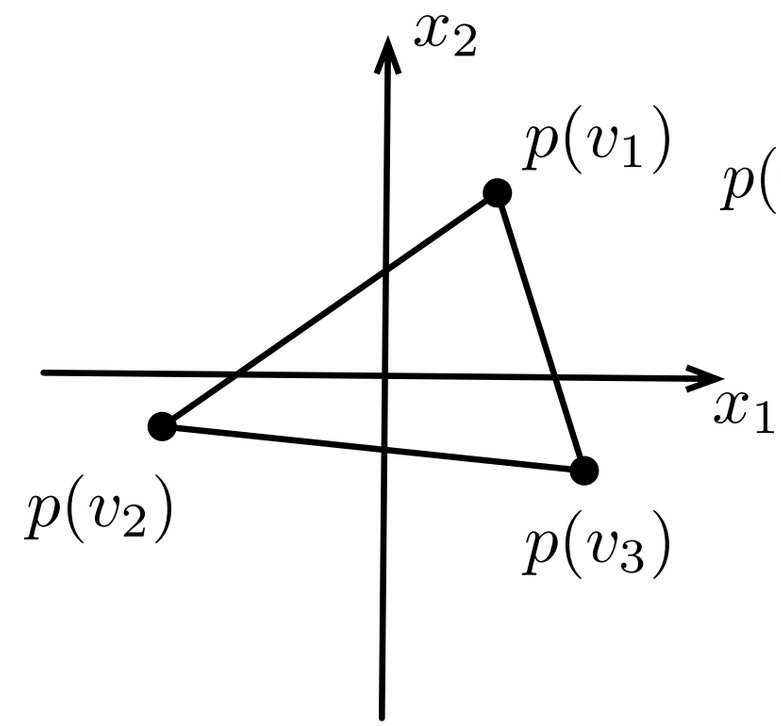
The Rigidity Matrix

infinitesimal motions define a system of equations...

$$(\xi(v_i) - \xi(v_j))^T (p(v_i) - p(v_j)) = 0$$

The Rigidity Matrix

$$R(p) \in \mathbb{R}^{|\mathcal{E}| \times 2|\mathcal{V}|}$$



$$p(v_i) = (p_i^x, p_i^y)$$

$$R(p) = \begin{bmatrix} p_1^x - p_2^x & p_1^y - p_2^y & p_2^x - p_1^x & p_2^y - p_1^y & 0 & 0 \\ p_1^x - p_3^x & p_1^y - p_3^y & 0 & 0 & p_3^x - p_1^x & p_3^y - p_1^y \\ 0 & 0 & p_2^x - p_3^x & p_2^y - p_3^y & p_3^x - p_2^x & p_3^y - p_2^y \end{bmatrix}$$

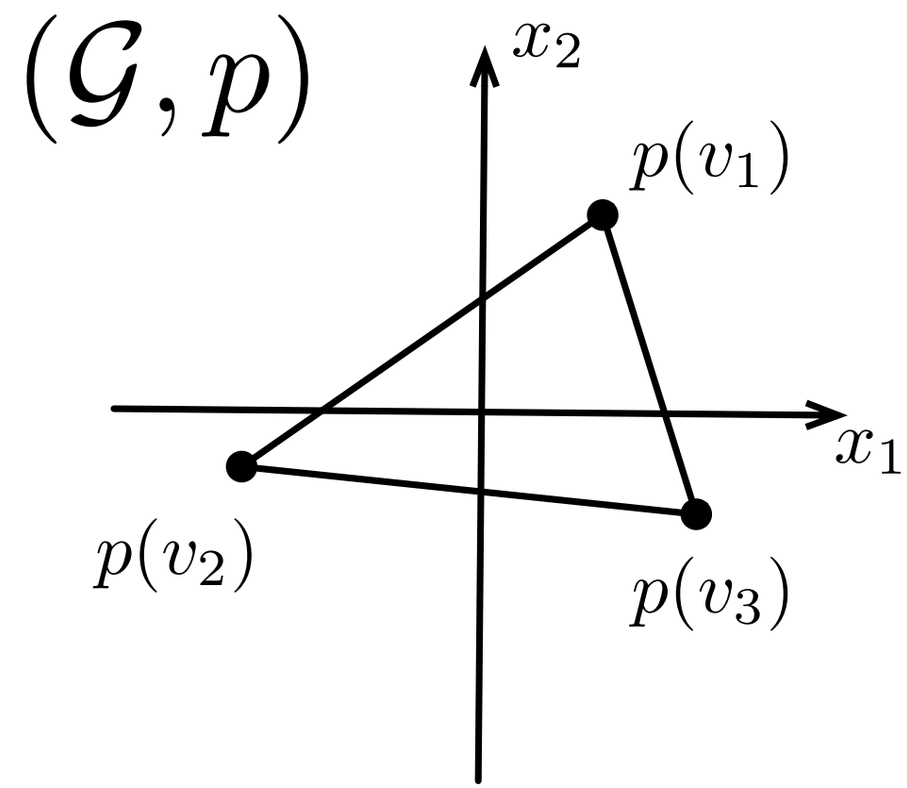
Graph Rigidity

another approach…

Edge 'Distance' Function

$$f(p) = \frac{1}{2}\begin{bmatrix} \vdots \\ \|p(v_i) - p(v_j)\|^2 \\ \vdots \end{bmatrix} \in \mathbb{R}^{|\mathcal{E}|}$$

$$\{v_i, v_j\} \in \mathcal{E}$$



the rigidity matrix is the 'linear' term in a Taylor series expansion of the edge function!

$$f(p + \delta_p) = f(p) + \frac{\partial f(p)}{\partial p}\delta_p + h.o.t.$$

The Rigidity Matrix

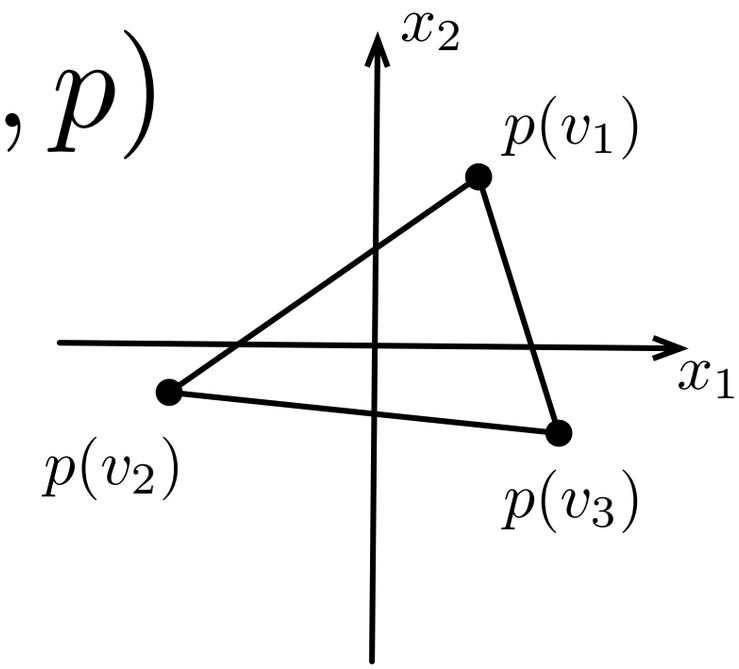
$$R(p) = \frac{\partial f(p)}{\partial p}$$

The Rigidity Matrix

The Rigidity Matrix

$$R(p) = \frac{\partial f(p)}{\partial p}$$

$(\mathcal{G}, p)$



Lemma 1 (Tay1984) *A framework $(\mathcal{G}, p)$ is infinitesimally rigid if and only if* $\mathbf{rk}[R] = 2|\mathcal{V}| - 3$

A framework is *minimally infinitesimally rigid* (MIR) if it is infinitesimally rigid and minimally rigid.

$$\Rightarrow \mathbf{rk}[R(p)] = 2|\mathcal{V}| - 3 = |\mathcal{E}|$$

MIR frameworks have ***full row rank***

The Formation Control Problem

Distance Potential

$$V(p) = \sum_{\{i,j\}\in\mathcal{E}} V_{ij}(\|p_i - p_j\|) = \frac{1}{4}\sum_{\{i,j\}\in\mathcal{E}} (\|p_i - p_j\|^2 - d_{ij}^2)^2$$

$$\begin{aligned}\dot{p}_i(t) &= -\frac{\partial}{\partial p_i} V(p) \\ &= -\sum_{\{i,j\}\in\mathcal{E}} \left(\|p_i(t) - p_j(t)\|^2 - d_{ij}^2\right)(p_i(t) - p_j(t))\end{aligned}$$

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

a Lyapunov approach

$$F(p) = \frac{1}{4}\sigma^T\sigma$$

$$\frac{d}{dt}F(p) = -\sigma^T R(p)R^T(p)\sigma \leq 0$$

$$\frac{d}{dt}F(p) = 0 \Leftrightarrow R^T(p)\sigma = 0$$

some notations…

$$e_{ij}(t) = e_k(t) = p_i(t) - p_j(t)$$

$$k = \{i, j\} \in \mathcal{E}$$

$$\sigma_k = \|e_k\|^2 - d_k^2 = e_k^T e_k - d_k^2$$

recall: the potential function defining a gradient dynamical system can serve as a Lyapunov function candidate!

Formation Control - Stability Analysis

$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$

Stability and full row rank

a rigidity matrix with full row rank (i.e. minimally infinitesimally rigid framework) $\Leftrightarrow \{\sigma \mid R^T(p)\sigma = 0\} = \{0\}$

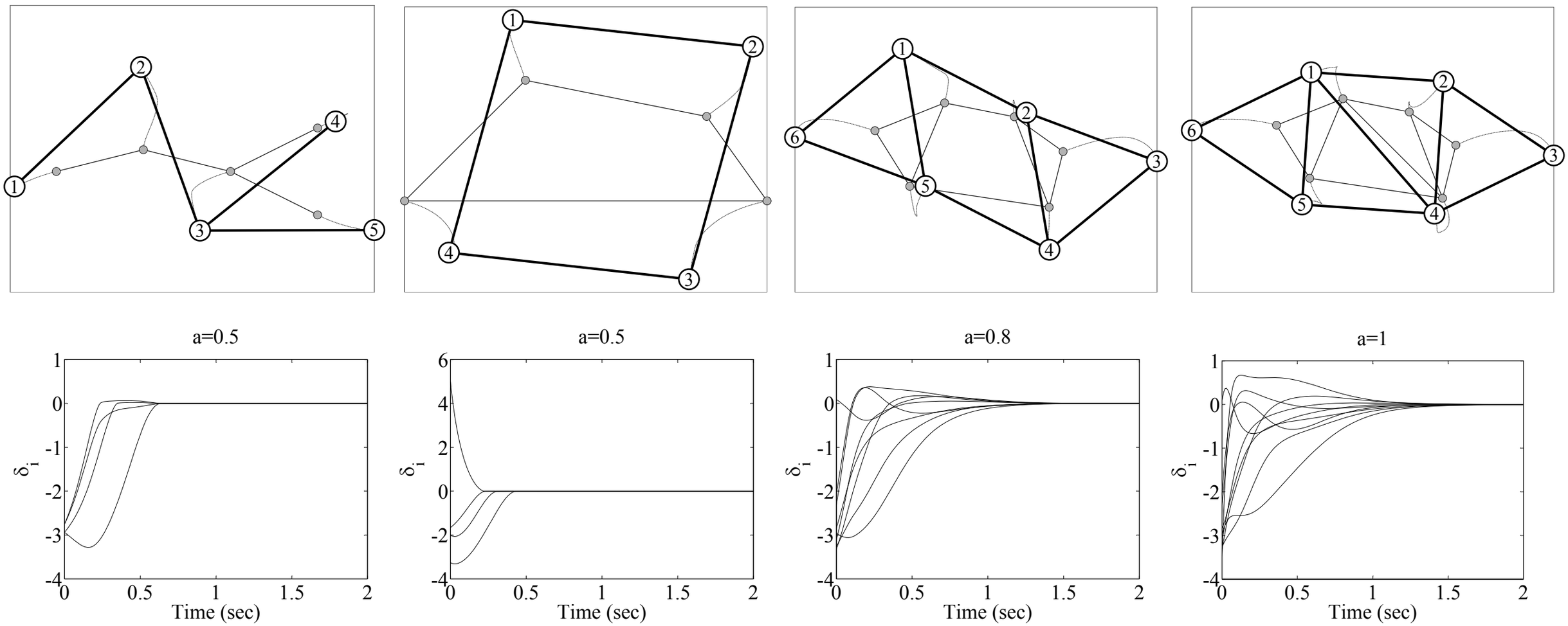
$$\frac{d}{dt}F(p) = -\sigma^T \underbrace{R(p)R^T(p)}_{\text{a positive definite matrix!}}\sigma \leq 0 \qquad \Rightarrow \frac{d}{dt}F(p) = 0 \Leftrightarrow \sigma = 0$$

Theorem

If the rigidity matrix has full row rank then the distributed distance-based formation control law (exponentially) converges to the specified formation set (locally).

Formation Control

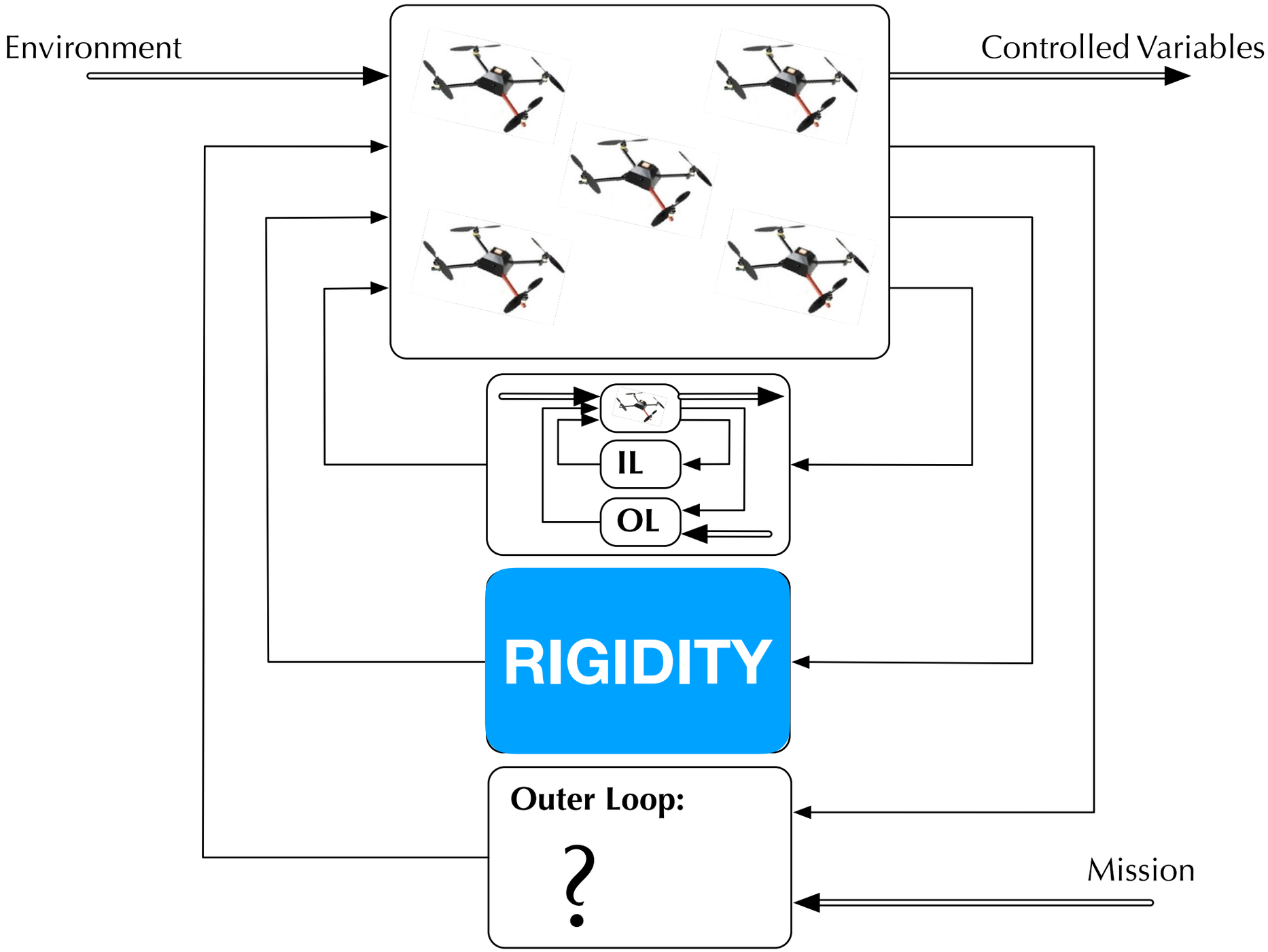
$$\dot{p} = -R^T(p)R(p)p - R^T(p)d^2$$



(a) Tree (b) Cycle (c) General flexible (d) MIR

Towards a Multi-Robot Control Architecture

what is the architecture for a *multi-robot* system?



Outlook and Challenges

Sensors

- relative position sensors
- distance sensing
- bearing sensing

Coordinate Frames

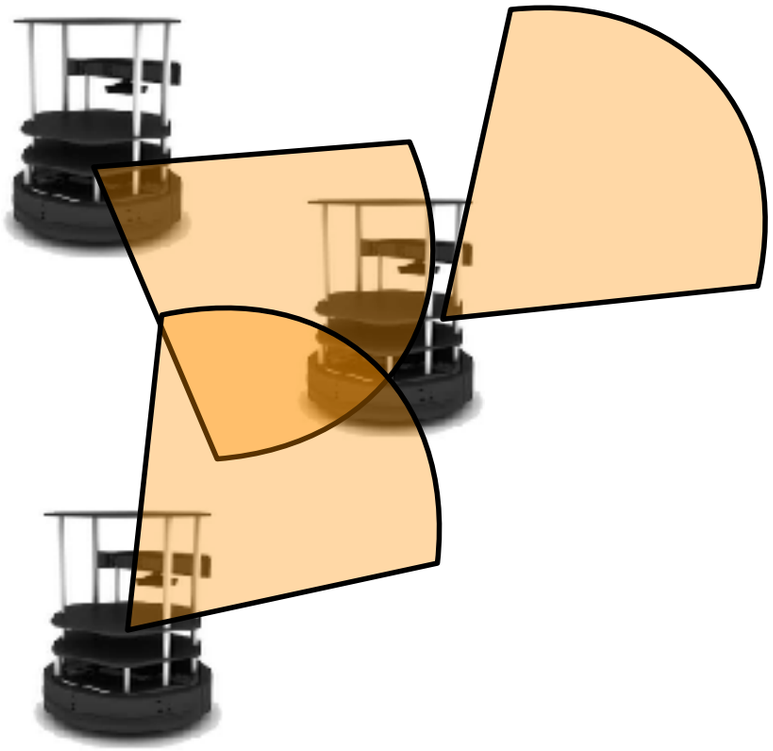
- common/global reference frames
- local/body frame

Symmetry

- undirected graphs
- directed graphs

Rigidity Theory
Bearing Rigidity Theory
SE(2) Rigidity Theory

measurement-independent theory?



Questions?

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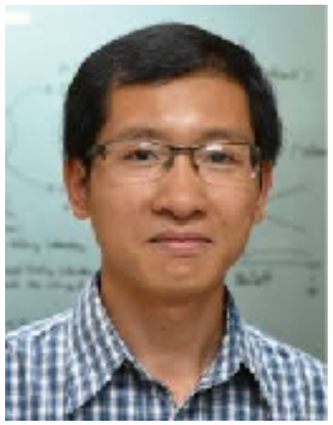

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