A physical interpretation of the rigidity matrix

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Towards the Discovery of Common Models and Methods

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Credits

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(D) Viet Hoang Pham
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3 The symmetric rigidity matrix
   - The symmetric rigidity matrix
   - Physical meaning of the eigenvectors
   - Further properties of the eigenvalues

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Multi-agent systems & Distributed formation control

Agents and multi-agent systems:
- An agent is understood as a dynamical system.
- A multi-agent system is a collection, a group, or a team of dynamical systems.

Distributed formation control:
- No centralized controller for a given multi-agent system.
- Each agent has its own controller based on interaction with its neighboring agents.
- Only the distances among agents are controlled by relative interactions; → but a formation defined w.r.t a global coordinate frame is achieved, upto translations and rotations.
Formation with Distance Constraints

- Only distances (edges) are constrained
- Formation is fixed (rigid) or not-fixed (flex)?
Formation with Distance Constraints

- Only distances are constrained
- Formation is fixed (rigid) or not-fixed (flex)?
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Formation with Distance Constraints

- Only distances are constrained
- Formation is fixed (rigid) or not-fixed (flex)?

Rigid graphs?

- Unique; but any point of epsilon neighborhood, the configuration is not unique! (infinitesimally rigid)
Consistency between the overall and the local tasks

If all the agents complete their local task, then the overall task is achieved? What condition is required for $G = (\mathcal{V}, \mathcal{E})$ in order to satisfy

$$\forall (i, j) \in \mathcal{E}, \|p_i - p_j\| = \|p_i^* - p_j^*\| \Rightarrow \forall i, j \in \mathcal{V}, \|p_i - p_j\| = \|p_i^* - p_j^*\|?$$

Equivalence (the local tasks)  Congruence (the overall task)

Rigidity or the persistence of $G$ specifying the minimum number and the distribution pattern of edges.

(E) Not rigid.  (F) Rigid.  (G) Not persistent.  (H) Persistent.
Use of Graph rigidity for Consistency in Task

Given an undirected graph $G = (V, E)$, where $V = \{1, \ldots, N\}$, let us assign $p_i \in \mathbb{R}^n$ to each vertex $i$ for all $i \in V$.

- Realization: $p = (p^T_1, \ldots, p^T_N)^T \in \mathbb{R}^{nN}$, Framework: $(G, p)$
- Equivalence: Two frameworks $(G, p)$ and $(G, q)$ are equivalent if
  \[ \forall (i, j) \in E, \|p_i - p_j\| = \|q_i - q_j\|. \]
- Congruence: Two frameworks $(G, p)$ and $(G, q)$ are congruent if
  \[ \forall i, j \in V, \|p_i - p_j\| = \|q_i - q_j\|. \]

**Definition (Rigidity)**

A framework $(G, p)$ is rigid if there exists a neighborhood $U_p$ of $p$ such that all frameworks equivalent to $(G, p)$ are congruent in $U_p$.

- If $(G, p)$ is rigid, then the overall task and the local tasks is consistent.
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   • Motivation and definition
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6 CONCLUSIONS
\[ \mathcal{F} = (\mathcal{G}, p): \text{a framework in } \mathbb{R}^2 \]

- \(\mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{V} = \{1, \ldots, n\}, \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}, |\mathcal{V}| = n, |\mathcal{E}| = m\)
- \(p_i = [x_i, y_i]^T, p = [p_1^T, \ldots, p_n^T]^T: \text{a realization in } \mathbb{R}^2\)
- \(H \in \mathbb{R}^{m \times n}: \text{the incidence matrix}\)

\[ H = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & 2 & 3 & 4 & 1
\end{bmatrix} \]

\(V = \{1, 2, 3, 4\}\)
\(E = \{(1,2),(1,3),(1,4),(2,3),(3,4)\}\)

\[ H \mathcal{F} = \mathcal{F}' \]
\[ \exists \zeta > 0 \text{ such that } \|p_i - p_j\| = \zeta \|p'_i - p'_j\|, \forall i, j \in \mathcal{V} \]

**Figure:** A framework with four vertices and five edges.
**Incidence rigidity & Similarity**

- \( \mathcal{F} = (\mathcal{G}, p) \): a framework in \( \mathbb{R}^2 \)
  - \( \mathcal{G} = (\mathcal{V}, \mathcal{E}), \mathcal{V} = \{1, \ldots, n\}, \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}, |\mathcal{V}| = n, |\mathcal{E}| = m \)
  - \( p_i = [x_i, y_i]^T, p = [p_1^T, \ldots, p_n^T]^T \): a realization in \( \mathbb{R}^2 \)

- \( \mathcal{F} = (\mathcal{G}, p) \) and \( \mathcal{F}' = (\mathcal{G}, p') \) are similar iff. \( \exists \zeta > 0 \) such that
  \[
  \|p_i - p_j\| = \zeta \|p'_i - p'_j\|, \quad \forall i, j \in \mathcal{V}
  \]

**\( \zeta \): the scale factor**

**Figure:** Similar frameworks: \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \).
Denote $z_{ij} = p_j - p_i$, $(i, j) \in E$: displacement vector

Labeling $m$ edges, we have $z = [z_1^T, \ldots, z_m^T]^T = (H \otimes I_2)p \in \mathbb{R}^{2m}$
Denote $z_{ij} = p_j - p_i$, $(i, j) \in \mathcal{E}$: displacement vector.

Labeling $m$ edges, we have $z = [z_1^T, \ldots, z_m^T]^T = (H \otimes I_2)p \in \mathbb{R}^{2m}$.

The distance function $f_G : \mathbb{R}^{2n} \to \mathbb{R}^m$,

$$f_G : p \mapsto [\|z_1\|^2, \ldots, \|z_m\|^2]^T.$$

The rigidity matrix $R := \frac{1}{2} \frac{\partial f_G(p)}{\partial p \in \mathbb{R}^{m \times 2n}}$. 
Denote $z_{ij} = p_j - p_i, \ (i,j) \in \mathcal{E}$: displacement vector

Labeling $m$ edges, we have $z = [z_1^T, \ldots, z_m^T]^T = (H \otimes I_2)p \in \mathbb{R}^{2m}$

The distance function $f_G : \mathbb{R}^{2n} \rightarrow \mathbb{R}^m$,

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The rigidity matrix $R := \frac{1}{2} \partial f_G(p)/\partial p \in \mathbb{R}^{m \times 2n}$

\begin{figure}[h]
\centering
\begin{align*}
R &= \begin{bmatrix}
(p_1 - p_2)^T & -(p_1 - p_2)^T & 0 & 0 \\
(p_1 - p_3)^T & 0 & -(p_1 - p_3)^T & 0 \\
(p_1 - p_4)^T & 0 & 0 & -(p_1 - p_4)^T \\
0 & (p_2 - p_3)^T & -(p_2 - p_3)^T & 0 \\
0 & 0 & (p_3 - p_4)^T & -(p_3 - p_4)^T
\end{bmatrix}
\end{align*}
\end{figure}

**Figure:** The rigidity matrix $R$. 
Denote $z_{ij} = p_j - p_i$, $(i, j) \in \mathcal{E}$: displacement vector

Labeling $m$ edges, we have $z = [z_1^T, \ldots, z_m^T]^T = (H \otimes I_2)p \in \mathbb{R}^{2m}$

The distance function $f_G : \mathbb{R}^{2n} \to \mathbb{R}^m$,

$$f_G : p \mapsto [\|z_1\|^2, \ldots, \|z_m\|^2]^T.$$

The rigidity matrix $R := \frac{1}{2} \partial f_G(p)/\partial p \in \mathbb{R}^{m \times 2n}$

A framework is \textit{infinitesimally rigid} iff. $\text{rank}(R) = 2n - 3$. 
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The symmetric rigidity matrix

$$M := R^T R.$$ (2)

where $R$ is the rigidity matrix.

**Figure:** The symmetric rigidity matrix $M$ resembles the graph Laplacian matrix.
The symmetric rigidity matrix

\[ M := R^T R. \]  \hspace{1cm} (2)

where \( R \) is the rigidity matrix. Some properties of \( M \):

- \( M \in \mathbb{R}^{2n \times 2n} \) is symmetric, positive semidefinite.
- \( \mathcal{N}(M) = \mathcal{N}(R) \), \( \text{rank}(M) \leq 2n - 3 \).
  \[ \Rightarrow \] \( M \) has at least three zero eigenvalues.
- \( \mathcal{F} \) is infinitesimally rigid iff. \( \text{rank}(M) = 2n - 3 \).
The symmetric rigidity matrix $M$ has

- Eigenvalues: $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$
- Eigenvectors: $\nu^1, \nu^2, \ldots, \nu^{2n}$, where $\nu^k = [(\nu^k_1)^T, \ldots, (\nu^k_n)^T]^T \in \mathbb{R}^{2n}$
The symmetric rigidity matrix $M$ has

- Eigenvalues: $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}, (\lambda_1 = \lambda_2 = \lambda_3 = 0)$
- Eigenvectors: $v^1, v^2, \ldots, v^{2n}$, where $v^k = [(v^k_1)^T, \ldots, (v^k_n)^T]^T \in \mathbb{R}^{2n}$

$v^1, v^2, v^3$ associate with three zero eigenvalues$^2$

$$
v^1 = [1 \ 0 \ 1 \ 0 \ \ldots \ 1 \ 0]^T,
$$

$$
v^2 = [0 \ 1 \ 0 \ 1 \ \ldots \ 0 \ 1]^T,
$$

$$
v^3 = [-y_1 \ x_1 \ -y_2 \ x_2 \ \ldots \ -y_n \ x_n]^T.
$$

$^2$Sun2015a.
The symmetric rigidity matrix $M$ has

- **Eigenvalues:** $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$
- **Eigenvectors:** $v^1, v^2, \ldots, v^{2n}$, where $v^k = [(v^k_1)^T, \ldots, (v^k_n)^T]^T \in \mathbb{R}^{2n}$

$v^1, v^2, v^3$ associate with three zero eigenvalues

\[
\begin{align*}
v^1 &= [1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0]^T, \\
v^2 &= [0 \quad 1 \quad 0 \quad 1 \quad \ldots \quad 0 \quad 1]^T, \\
v^3' &= v^3 + \tilde{y} v^1 - \tilde{x} v^2,
\end{align*}
\]

where $\tilde{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $\tilde{y} = \frac{1}{n} \sum_{i=1}^{n} y_i$. 
The symmetric rigidity matrix $M$ has

- **Eigenvalues**: $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, $(\lambda_1 = \lambda_2 = \lambda_3 = 0)$
- **Eigenvectors**: $v^1, v^2, \ldots, v^{2n}$, where $v^k = [(v^k_1)^T, \ldots, (v^k_n)^T]^T \in \mathbb{R}^{2n}$

$v^1, v^2, v^3$ associate with three zero eigenvalues

$v^1 = [1 \ 0 \ 1 \ 0 \ \ldots \ 1 \ 0]^T, \quad v^2 = [0 \ 1 \ 0 \ 1 \ \ldots \ 0 \ 1]^T, \quad v^3' = v^3 + \bar{y}v^1 - \bar{x}v^2,$

**Figure**: $v^1, v^2, v^3'$ correspond to infinitesimally rigid motions.
The symmetric rigidity matrix $M$ has

- **Eigenvalues:** $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, ($\lambda_1 = \lambda_2 = \lambda_3 = 0$)
- **Eigenvectors:** $v^1, v^2, \ldots, v^{2n}$, where $v^k = [(v^1_k)^T, \ldots, (v^n_k)^T]^T \in \mathbb{R}^{2n}$

$v^1, v^2, v^3$ associate with three zero eigenvalues

$$v^1 = [1 \ 0 \ 1 \ 0 \ \ldots \ 1 \ 0]^T,$$
$$v^2 = [0 \ 1 \ 0 \ 1 \ \ldots \ 0 \ 1]^T,$$
$$v^3' = v^3 + \bar{y}v^1 - \bar{x}v^2,$$

For $\lambda_k \neq 0 \Rightarrow$ physical interpretation of $v^k$?
Statics of Frameworks

Framework: rods and joints model\textsuperscript{2}

\textsuperscript{2}Roth1981.
Framework: rods and joints model\(^2\)

- Stress: a set of scalars \( w = [w_{ij}]_{(i,j) \in \mathcal{E}} \) defined for each edge
- Equilibrium stress:
  \[
  \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j) = 0, \quad \forall i = 1, \ldots, n. \tag{3}
  \]
  
- A stress is *trivial* when \( w_{ij} = 0, \forall (i, j) \in \mathcal{E} \).
- Stress free: Only the trivial makes the equilibrium stress be satisfied
- Rigid graphs: Stress free
- Minimally rigid graphs: Stress free & All edges linearly independent (in the sense of rigidity matrix)
- Flex graphs: Edges linearly dependent (i.e., non-trivial \( w_{ij} \) makes the equilibrium stress be satisfied)

\(^2\)Roth1981.
Framework: rods and joints model\(^3\)

\[^3\text{Roth1981.}\]
Framework: rods and joints model

- **Stress**: a set of scalars \( w = [w_{ij}]_{(i,j) \in \mathcal{E}} \) defined for each edge

- **Equilibrium stress**:
  \[
  \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j) = 0, \quad \forall i = 1, \ldots, n. \tag{4}
  \]

- A stress is *trivial* when \( w_{ij} = 0, \forall (i, j) \in \mathcal{E} \).

- **Stress free**: Only the trivial makes the equilibrium stress be satisfied

- **Rigid graphs**: Stress free

- **Minimally rigid graphs**: Stress free & All edges linearly independent (in the sense of rigidity matrix)

- **Flex graphs**: Edges linearly dependent (i.e., non-trivial stress makes the equilibrium stress be satisfied)

\(^3\text{Roth1981.}\)
Statics of Frameworks

- $F = [F_1^T, \ldots, F_n^T]^T \in \mathbb{R}^{2n}$ is an equilibrium force if

$$
\sum_{i=1}^{n} F_i = 0, \tag{5}
$$

$$
\sum_{i=1}^{n} p_i \times F_i = 0, \tag{6}
$$

where $\times$ denotes the cross product.

[Diagram of a framework with forces $F_1$, $F_2$, and $F_3$ applied at nodes 1, 2, and 3, respectively.]
$F = [F_1^T, \ldots, F_n^T]^T \in \mathbb{R}^{2n}$ is an equilibrium force if

$$
\sum_{i=1}^{n} F_i = 0, \quad (5)
$$

$$
\sum_{i=1}^{n} p_i \times F_i = 0, \quad (6)
$$

where $\times$ denotes the cross product.

$F$ is resolvable if $\exists$ scalars $w_{ij}$ s.t.

$$
F_i + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j) = 0, \quad (7)
$$

for all $i = 1, \ldots, n$. 

For each eigenvector $\mathbf{v}^k = [v_1^k, \ldots, v_n^k]^T$, 

$$M\mathbf{v}^k = R^T R\mathbf{v}^k = \lambda_k \mathbf{v}^k$$

Let $\mathbf{w} = [w_{ij}]_{(i,j)\in\mathcal{E}} = R\mathbf{v}^k \in \mathbb{R}^m$, 

$$M\mathbf{v}^k = R^T (R\mathbf{v}^k) = R^T \mathbf{w} = \lambda_k \mathbf{v}^k, \quad k = 1, \ldots, 2n. \quad (8)$$

or $-\lambda_k v_i^k + \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j) = 0, \quad i = 1, \ldots, n; k = 1, \ldots, 2n. \quad (9)$
For each eigenvector \( v^k = [v_1^k, \ldots, v_n^k]^T \),

\[
M v^k = R^T R v^k = \lambda_k v^k
\]

Let \( w = [w_{ij}]_{(i,j) \in \mathcal{E}} = R v^k \in \mathbb{R}^m \),

\[
M v^k = R^T (R v^k) = R^T w = \lambda_k v^k, \quad k = 1, \ldots, 2n. \quad (8)
\]

or \(-\lambda_k v_i^k + \sum_{j \in \mathcal{N}_i} w_{ij} (p_i - p_j) = 0, \quad i = 1, \ldots, n; k = 1, \ldots, 2n. \quad (9)\]

**Theorem**

Given an infinitesimally rigid framework \( \mathcal{F} \) in a plane. Then each vector \( F = -\lambda_k v^k \) \((k = 4, \ldots, 2n - 3)\) is a resolvable force, where \( v^k \) is the eigenvector corresponding to a nonzero eigenvalue of the symmetric rigidity matrix \( M \) of \( \mathcal{F} \).
### Physical Meanings of Eigenvectors of $\mathbf{M}$

<table>
<thead>
<tr>
<th>Physical interpretations</th>
<th>Example: equilateral triangle framework</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu^1, \nu^2, \nu^3$</td>
<td><img src="image" alt="Figure" /></td>
</tr>
<tr>
<td>- Infinitesimally rigid motions</td>
<td></td>
</tr>
<tr>
<td>- Trivial stresses</td>
<td><img src="image" alt="Figure" /></td>
</tr>
<tr>
<td>$\nu^4, \nu^5, \ldots, \nu^{2n}$</td>
<td><img src="image" alt="Figure" /></td>
</tr>
<tr>
<td>- Equilibrium forces &amp; Resolvable forces</td>
<td><img src="image" alt="Figure" /></td>
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</tbody>
</table>

**Figure:** The eigenvectors of an equilateral triangular frameworks
Further Properties of Eigenvalues of Matrix $M$

<table>
<thead>
<tr>
<th>Properties</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Invariant to</td>
<td></td>
</tr>
<tr>
<td>• translation</td>
<td><img src="image1" alt="F" /></td>
</tr>
<tr>
<td>• rotation</td>
<td><img src="image2" alt="F'" /></td>
</tr>
<tr>
<td>• reflection</td>
<td><img src="image3" alt="F”" /></td>
</tr>
<tr>
<td>$\lambda_i = \lambda_i', \ \forall i = 1,\ldots,2n.$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Properties</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proportional to</td>
<td></td>
</tr>
<tr>
<td>• (scale factor)$^2$</td>
<td><img src="image4" alt="F" /></td>
</tr>
<tr>
<td>$\lambda_i = \zeta^2 \lambda_i', \ \forall i = 1,\ldots,2n.$</td>
<td><img src="image5" alt="F'" /></td>
</tr>
<tr>
<td></td>
<td><img src="image6" alt="F”" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Properties</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increasing when adding more edges</td>
<td><img src="image7" alt="F" /></td>
</tr>
<tr>
<td>$\lambda_i \leq \lambda_i', \ \forall i = 1,\ldots,2n.$</td>
<td><img src="image8" alt="F’" /></td>
</tr>
</tbody>
</table>

**Figure:** $F$ has matrix $M$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$, and $F'$ has matrix $M'$ with eigenvalues $\lambda'_1 \leq \lambda'_2 \leq \ldots \leq \lambda'_{2n}$.
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6 Conclusions
Consider the triangular framework depicted in the case (d) in page 13 (i.e., $k = 4$).

- Stress forces along the edges $\sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$

- Forces to each nodes $-\lambda_4 v^4_i$

Each node is governed by double integrator dynamics (i.e., $\ddot{p}_i = u_i$).

1) $u_i = -\lambda_4 v^4_i + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$

2) $u_i = -\lambda_4 v^4_i(0) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$

3) $u_i = -\lambda_4 \text{normalized}(v^4_i(0) + 0.1 q^4_i(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$, where $(v^4_i)^T q^4_i = 0$

4) $u_i = -\lambda_4 \text{normalized}(v^4_i(0) + 0.5 q^4_i(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$

5) $u_i = -\lambda_4 \text{normalized}(v^4_i(0) + 1.0 q^4_i(0)) + \sum_{j \in \mathcal{N}_i} w_{ij}(p_i - p_j)$
Case 1

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory
Case 2

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory

Graph showing the trajectories of three agents in a 2D coordinate system.
Case 3

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory

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Case 4

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory

$x$ vs. $y$ plot for trajectories of three agents.
Case 5

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory
Each node is governed by double integrator dynamics (i.e., $\ddot{p}_i = u_i$).

6) $u_i = -\lambda_4(0)v_i^4(0) + \sum_{j \in N_i} w_{ij}(p_i - p_j)$

7) $u_i = -\lambda_4(0)v_i^4(0) + \sum_{j \in N_i} w_{ij}(0)(p_i - p_j)$

8) $u_i = -\lambda_4(0)\text{normalized}(v_i^4(0) + 0.1q_i^4(0)) + \sum_{j \in N_i} w_{ij}(p_i - p_j)$, where

$$ (v_i^4)^T q_i^4 = 0 $$

9) $u_i = -\lambda_4(0)\text{normalized}(v_i^4(0) + 0.5q_i^4(0)) + \sum_{j \in N_i} w_{ij}(p_i - p_j)$

10) $u_i = -\lambda_4(0)\text{normalized}(v_i^4(0) + 1.0q_i^4(0)) + \sum_{j \in N_i} w_{ij}(p_i - p_j)$
Case 6

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory
Case 7

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory
Case 8

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory

y
-1 -0.5 0 0.5 1 1.5
x
-1 -0.5 0 0.5 1 1.5
Case 9

Trajectories of three agents

- Initial positions
- Final positions
- Trajectory
Case 10
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Motivation

$\lambda_4(M)$ is usually used as a rigidity index \( \lambda_4 > 0 \Leftrightarrow \mathcal{F} \) is infinitesimally rigid \( \lambda_4 \) depends quadratically on the scale factor \( \zeta \).

$\Rightarrow$ cannot compare rigidity between different frameworks.

---

\(^4\text{Zelazo2012.}\)
Consider a framework $\mathcal{F} = (G, p)$ in the plane,

**Definition**

The **worst-case rigidity index** of $\mathcal{F}$ is defined as

$$\chi = \frac{\lambda_4}{\sum_{i=1}^{2n} \lambda_i} = \frac{\lambda_4}{\text{tr}(M)}. \quad (8)$$

**Definition**

The **imbalance index** of the framework $\mathcal{F}$ is defined as

$$\xi = \frac{\lambda_4}{\lambda_{2n}}. \quad (9)$$
**Proposition**

Assume $F_1 = (G, p)$ and $F_2 = (G, p')$ are two similar frameworks with the worst-case rigidity indices $\chi_1, \chi_2$ and the imbalance indices $\xi_1, \xi_2$. Then $\chi_1 = \chi_2$ and $\xi_1 = \xi_2$.

The new rigidity indices:
- $\chi > 0$ and $\xi > 0 \iff F$ is infinitesimally rigid
- scale-free
- depend only on the framework’s shape.

**Figure:** $F_1, F_2$ and $F_3$ have the same worst-case rigidity index: $\chi_1 = \chi_2 = \chi_3$. 
Example 1: Triangular frameworks

<table>
<thead>
<tr>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>0.0508</td>
<td>0.0853</td>
<td>0.1585</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.0726</td>
<td>0.1284</td>
<td>0.2679</td>
</tr>
</tbody>
</table>
Example 2: Square frameworks

<table>
<thead>
<tr>
<th></th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi$</td>
<td>0</td>
<td>0.0253</td>
<td>0.0637</td>
<td>0.1250</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0</td>
<td>0.0627</td>
<td>0.1459</td>
<td>0.3333</td>
</tr>
</tbody>
</table>
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6 CONCLUSIONS
Main results:

- Further analysis on the symmetric rigidity matrix $M$:
  - Physical interpretation of the eigenvectors
  - Further properties of the eigenvalues
- Two scale-free rigidity indices:
  - The worst-case rigidity index $\chi$
  - The imbalance index $\xi$.

Further studies:

- Find more properties of the rigidity indices
- Relationship between the rigidity matrix and the stiffness matrix
- Extend the results to 3D frameworks.
Thank you!