Structural topology, singularity, and kinematic analysis

J-P. Merlet
HEPHAISTOS project
INRIA Sophia-Antipolis
Parallel robots

Definitions: a closed-loop mechanism whose end-effector is linked to the base by several independent kinematic chains.

early prototype: Gough, 1947
Parallel robots

Stewart, 1965: proposal for flight simulators, Gough was reviewer

[Diagram showing parallel robot with joints, moving platform, and ball-and-socket joint marked]
Parallel robots

Why is this interesting?:

• excellent load/mass ratio: for serial at best 0.2, parallel robot may reach 10
• good to excellent accuracy
• good to very good rigidity

Drawback: limited workspace. But this may change if rigid legs are substituted by cables
Parallel robots
• **output**: the pose of the platform that may be parametrized by a set of parameters $X$

• **input**: the lengths $\rho$ of the legs

**Kinematics**: the relation between $X$ and $\rho$
Kinematics

- **Inverse kinematics**: \( X \rightarrow \rho \\
- **Direct kinematics**: \( \rho \rightarrow X \\

Inverse kinematics:

- coordinates of the \( A_i \) are known
- coordinates of \( C \) are known
- \( R \): rotation matrix
- \( CB = RCB_r \)
- \( AB = AO + OC + CB \)
- \( ||AB||^2 = \rho^2 \)
Direct kinematics

Constraints: \[ ||AB(X)||^2 = F(X) = \rho^2 \]

- a **square** system of non-algebraic or non-algebraic equations
- admits, **in general**, a finite number of solutions: up to 40 for the Gough platform
- efficient algorithms for finding **all** solutions: Groebner basis, interval analysis, elimination
- **20 years of work**

**Note:** direction of the leg (aka bearing) may be used as additional constraints for the DK
Direct kinematics

admits, in general, a finite number of solutions

⇓

for fixed values of $\rho$ there are no motion of the platform

in general: assume that you have found a solution $X_0$, then there is no other solution in the neighborhood of $X_0$

Rank theorem: if $J^{-1} = \partial F / \partial X$ has full rank, then there is no other solution in the neighborhood of $X_0$
Direct kinematics

Infinite rigidity unless $J^{-1}$ is singular

If $J^{-1}$ is singular

• platform will move although the leg lengths are fixed
• loss of rigidity
• loss of controlability
• the infinitesimal motion of the platform may lead to another singularity $\Rightarrow$ finite motion
Variational analysis

Time-derivative of the previous system

\[ \dot{\rho} = J^{-1} \dot{X} \quad \Delta \rho = J^{-1} \Delta X \]

This is not velocity relation as \( \dot{X} \) is not a representative of the angular velocity of the platform.

If \( J^{-1} \) is singular there are \( \dot{X} \neq 0 \) such that \( \dot{\rho} = 0 \)

\[ \Downarrow \]

- infinitesimal motion of the platform for fixed leg lengths
- loss of control and rigidity
- close to where \( J^{-1} \) is singular: large amplification factor between the change in leg lengths and the amplitude of the platform motion
Statics analysis

- $\tau$: force in the leg (directed along the vector $\mathbf{AB}$)
- $\mathcal{F}, \mathcal{M}$: forces/torques acting on the platform

Mechanical equilibrium:

- $\sum \tau_i \frac{\mathbf{A}_i \mathbf{B}_i}{\rho_i} = (\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z)^T$
- $\sum \tau_i \frac{\mathbf{C} \mathbf{B}_i \times \mathbf{A}_i \mathbf{B}_i}{\rho_i} = (\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)^T$

In matrix form:

$$(\mathcal{F}, \mathcal{M})^T = U(X)^T \tau$$

with the i-th row $U_i$ of $U$:

$$U_i = \left( \frac{\mathbf{A}_i \mathbf{B}_i}{\rho_i}, \frac{\mathbf{C} \mathbf{B}_i \times \mathbf{A}_i \mathbf{B}_i}{\rho_i} \right)$$
Statics analysis

$$(\mathcal{F}, \mathcal{M})^T = \mathbf{U} (\mathbf{X})^T \tau$$

for a given pose $\mathbf{X}$ we have here a linear system in the $\tau$

$$\downarrow$$

$$\tau_i = \frac{|M_i|}{|U|}$$

everything is fine unless $|U| = 0 \Rightarrow \tau_i \to \infty$$
Statics analysis

\[ U(X) = H(X)J^{-1}(X) \]

where \( H \) is a non-singular matrix that is dependent only on the choice of the parameters for representing the orientation of the platform.

Hence: \( U \) singular \iff \( J^{-1} \) singular

Poses \( X \) where \( U \) is singular are called singular poses of the parallel robot.
Singularity analysis

Why studying singularities?

Drawbacks:

• loss of control

• possible breakdown of the robot due to large forces in the leg

Advantages

• high sensitivity of the $\tau$ with respect to $\mathcal{F}, \mathcal{M}$: force/torque sensor
Singularity analysis

What is the problem?

- we have a matrix $U(X)$ in analytical form
- singularity are obtained for $X$ such that $|U(X)| = 0$

\[ \downarrow \]

Just compute $|U(X)| = 0$ and solve . . .

But

- $|U(X)|$ is a huge expression
- and solving if in $X$ is not a trivial task

We need tools and methodologies
Singularity analysis

Historical background

• In 1645, Sir Christopher Wren shows that curved surface may be built with line generators

• In 1813 Cauchy shows that the singularity of an articulated octahedron could be obtained only for concave configurations

• In 1908 the subject of the Prix Vaillant from the Academy of Sciences was to determine under which conditions a parallel mechanism may exhibit finite motion (prize won by Borel and Bricard)
Plücker vectors

Let $\mathcal{L}$ be a line in space and select two points $M_1, M_2$ on this line.

Let $P$ be the 6-dimensional vector

$$P = \begin{pmatrix}
\frac{M_1M_2}{||M_1M_2||} \\
\frac{OM_1 \times OM_2}{||M_1M_2||}
\end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$
Plücker vectors

\( P \) is the normalized Plücker vector of the line \( \mathcal{L} \)

- a line has a unique normalized Plücker vector
- let \( P_1, P_2 \) be the Plücker vectors of two lines \( \mathcal{L}_1, \mathcal{L}_2 \). These two lines will meet iff \( p_1 \cdot q_2 + q_1 \cdot p_2 = 0 \)

Remember

\[
U_i = \left( \frac{A_iB_i}{\rho_i}, \frac{C B_i \times A_iB_i}{\rho_i} \right)
\]

\( U_i \) is the Plücker vector of leg \( i \)

if \( U \) is singular, then we have a linear dependency between the Plücker vectors of the legs
Grassmann geometry

- the rank of a variety spanned by a set of $n$ Plücker vectors if the rank of $U$
- the rank cannot be greater than 6
- if $U$ is singular then the rank will be lower than 6

A singularity will occur only for certain geometrical configurations of the robot that may be determined using Grassmann geometry
# Grassmann geometry

**Variety** with rank from 1 to 3

<table>
<thead>
<tr>
<th>rank</th>
<th>a)</th>
<th>b)</th>
<th>c)</th>
<th>d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>2</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
<td><img src="image7" alt="Diagram" /></td>
<td><img src="image8" alt="Diagram" /></td>
</tr>
<tr>
<td>1</td>
<td><img src="image9" alt="Diagram" /></td>
<td><img src="image10" alt="Diagram" /></td>
<td><img src="image11" alt="Diagram" /></td>
<td><img src="image12" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Grassmann geometry

Use example

- three Plücker vectors $P_1, P_2, P_3$ span a variety of dimension $< 3 \Rightarrow$ singularity
- **Grassmann**: this occurs only if the lines lie in the same plane and meet at a common point
- writing this condition as function of $X$ gives the only singularity condition
Grassmann geometry

Variety with rank from 4 to 5

- Grassmann geometry gives a set of geometrical condition for which a singularity will occur
- each such condition can be expressed as function of $X$ (factorization of $|J^{-1}|$)
- plugging in the singularity condition in $J^{-1}$ and looking at the kernel of this matrix provides the nature of the infinitesimal motion
Grassmann geometry

All lines in a plane or intersecting the same point in this plane
Grassmann geometry

All lines intersecting the same line

\[ x_0 = 0.0835 \]
\[ y_0 = 0.437 \]
\[ z_0 = 20 \]

\[ \psi = 48^\circ \]
\[ \theta = -54^\circ \]
\[ \phi = 18^\circ \]
Tools: singularity detection

A very important practical problem

- determine if a singularity exists within a given workspace for $X$
- binary answer: yes or no
- if a singularity exists we are not interested in its location

Classical approach

- use optimization (e.g. find $X$ that minimize $|U(X)|^2$
- unsafe, slow
Tools: singularity detection

To solve efficiently this problem we will use interval analysis.

Interval $\mathcal{X} = [x, \bar{x}]$

Assume that we have

- a function $F(x)$
- a range $\mathcal{X}$ for $x$

interval evaluation of $F$ when $x \in \mathcal{X}$: a range $[A, B]$ such that

$$\forall \ x \in \mathcal{X} \text{ we have : } A \leq F(x) \leq B$$
Tools: singularity detection

How to construct an interval evaluation? the simplest one is the **natural evaluation**: substitute each mathematical operator by its interval equivalent.

**Example:**

\[ f(x) = x + \sin(x), \; x \in [1.1, 2] \]

\[
F([1.1, 2]) = [1.1, 2] + \sin([1.1, 2])
\]

\[
= [1.1, 2] + [0.8912, 1] = [1.9912, 3] = [A, B]
\]
Tools: singularity detection

Properties

• interval equivalent for any mathematical operators: no limitation to algebraic equations
• can be implemented to take into account round-off errors: numerically robust
• if $0 \not\in [\underline{F}, \overline{F}]$, then there is no $x$ in $\mathcal{X}$ that cancel $F$
• extrema of $F$ are bounded by the values of $A, B$
Tools: singularity detection

Bounded workspace: $\mathcal{W}$ (defined by a set of intervals for $X$, defining a box in a $m$ dimensional space)

Ingredients:

- choose an arbitrary point $X_0$ in $\mathcal{W}$, compute $|U(X_0)|$ with interval arithmetic, determine its sign (say $> 0$)

- $L$: a list of $n$ boxes $B_j$ with initially $n = 1$, $L = \{\mathcal{W}\}$

If we find a pose $X_1$ such that $|U(X_1)| < 0$, then $\mathcal{W}$ includes at least one singularity
Tools: singularity detection

Algorithm set $j = 1$

1. if $j > n$, then exit, **NO SINGULARITY**

2. compute the interval evaluation of $|U(B_j)| = [A_j, B_j]$

3. if $A_j > 0$, then $j = j + 1$, go to 1

4. if $B_j < 0$, then exit, **SINGULARITY**

5. if $A_j < 0$ and $B_j > 0$, then split $B_j$ into 2 boxes that are added to $L$, $j = j + 1$, $n = n + 2$, go to 1

**Very fast algorithm**

**Note:** uncertainties in the geometry of the robot may be taken into account to ensure that the real robot is singularity-free
Methodology: singularity index

Several indexes have been proposed to characterize the closeness to a singularity

- condition number of $\mathbf{U}$
- dexterity
- ...

But all of them have drawbacks and none of them have a physical meaning
Methodology: singularity index

Another approach: investigate the workspace of the robot such that all $\tau$ satisfy $|\tau_i| \leq \tau_{max}$

Finding this workspace:

- analytically for simple robot
- interval analysis for more complex robot
Methodology: singularity index

2 dof simple robot

\[ \tau_i = \frac{|M_i|}{|U|} \]

On the border of the workspace we have for at least one leg

- \[ \tau_{max} |U| = |M_i| \]
- or \[ \tau_{max} |U| = |M_i| \]

These equations define curves in the plane
Methodology: singularity index
Methodology: singularity index

A simple algorithm allows one to find the workspace border
Methodology: singularity index

Complex robot: cross-section for a 6 dof robot by using interval analysis
Conclusion

- singularity are relatively well mastered for parallel robot in terms of analysis for a given architecture
- analysis may take into account uncertainties in the geometry
- weaker for general analysis of mechanical architectures
- prospective:
  - bearing may be measured to simplify the direct kinematics: but measured with uncertainties
  - cable instead of rigid leg: they can only pull
  - cable with deformation (elasticity, sagging)