**Structural topology, singularity, and kinematic analysis**

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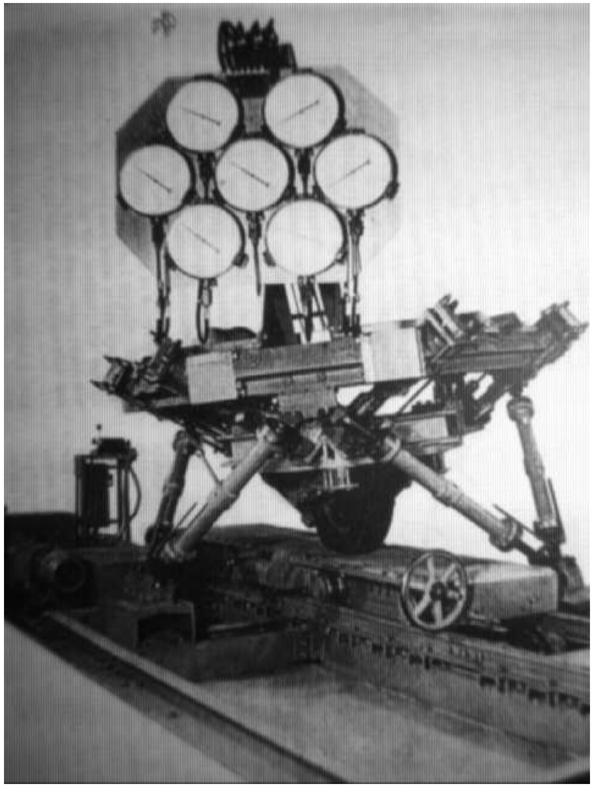
# Parallel robots

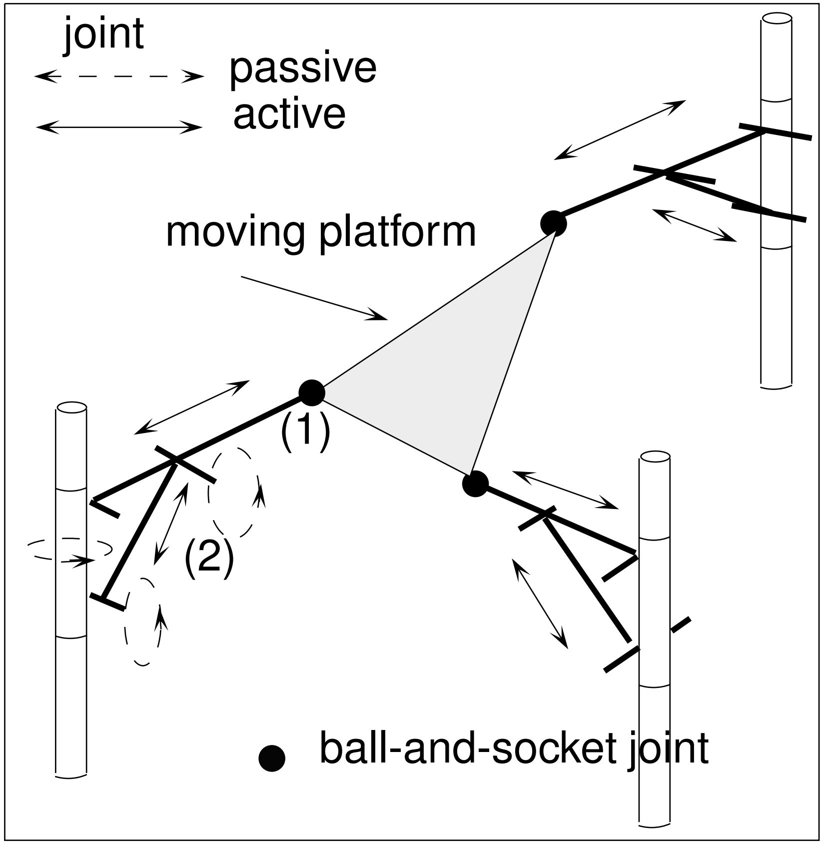
Definitions: *a closed-loop mechanism whose end-effector is linked to the base by several independent kinematic chains.*

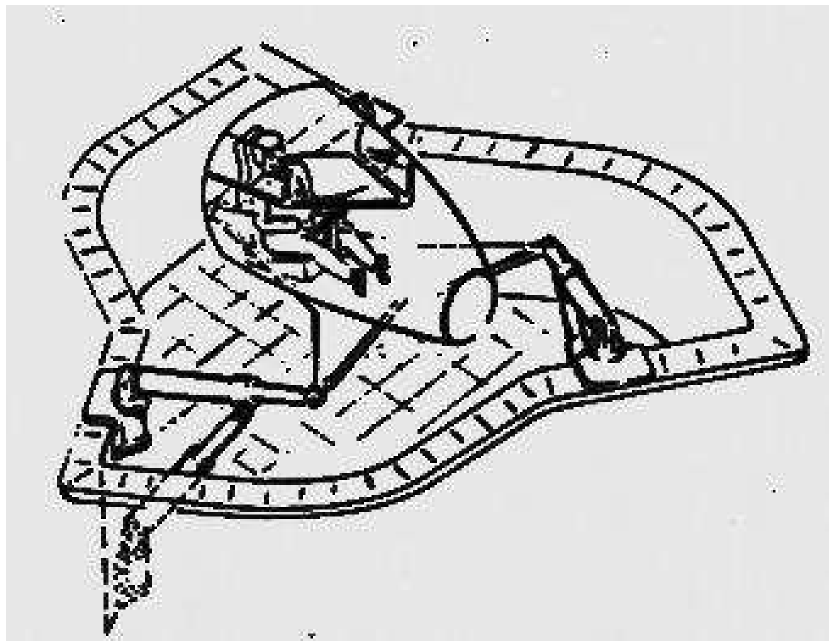
early prototype: Gough, 1947



# Parallel robots

Stewart, 1965: proposal for flight simulators, Gough was reviewer



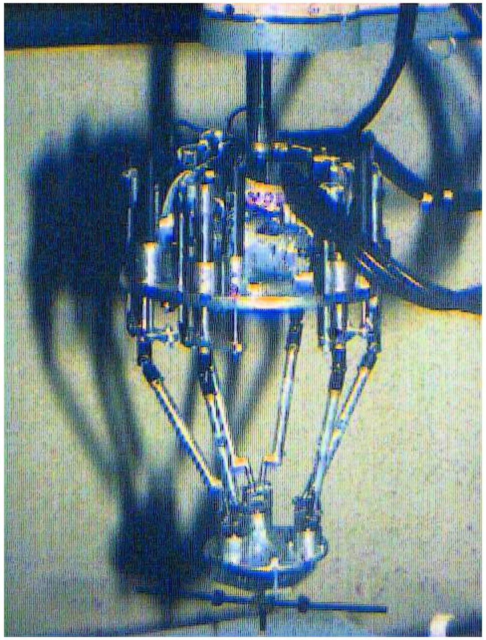
# Parallel robots

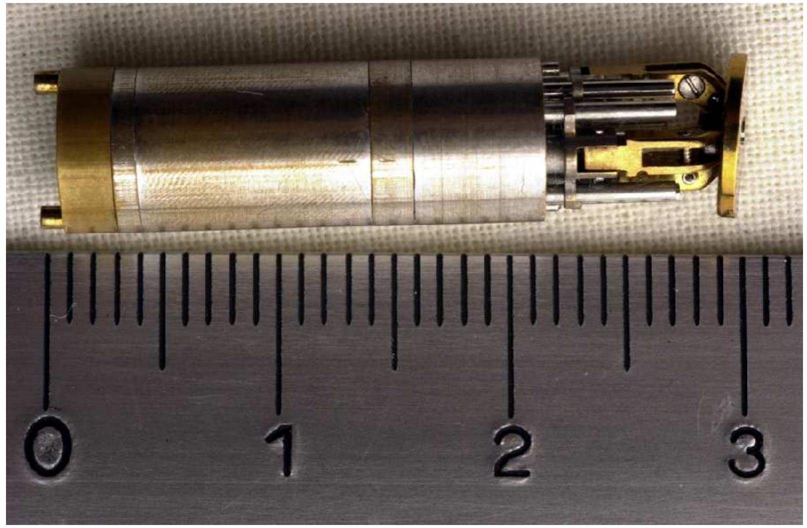
Why is this interesting?:

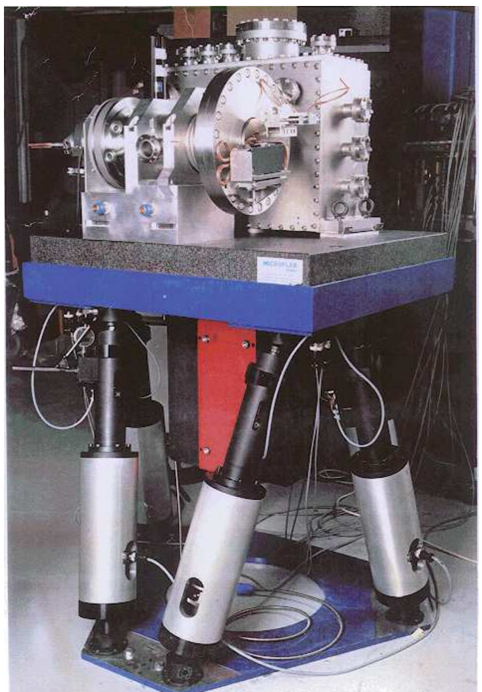
- excellent load/mass ratio: for serial at best 0.2, parallel robot may reach 10
- good to excellent accuracy
- good to very good rigidity

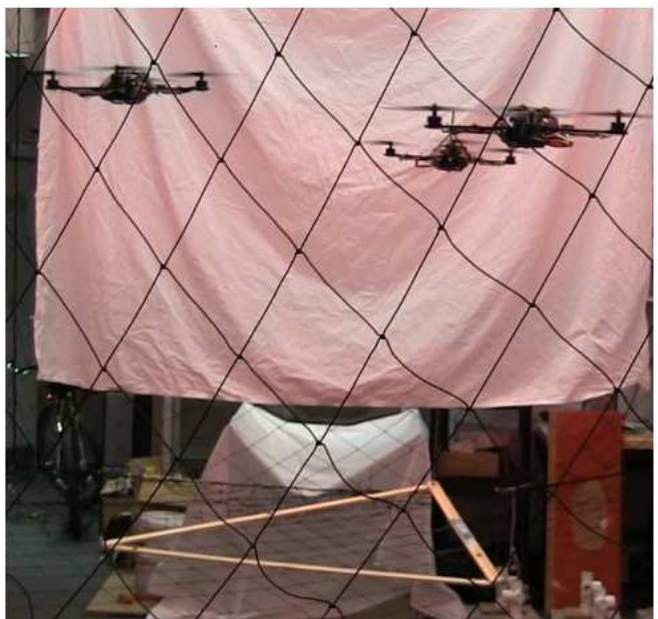
Drawback: limited workspace. But this may change if rigid legs are substituted by cables

# Parallel robots

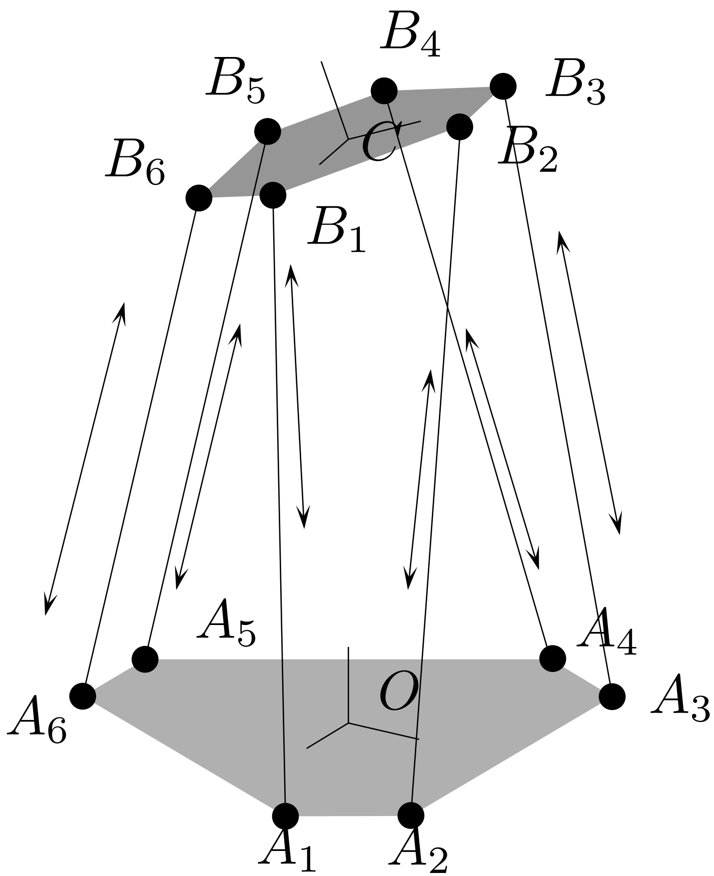






# Kinematics



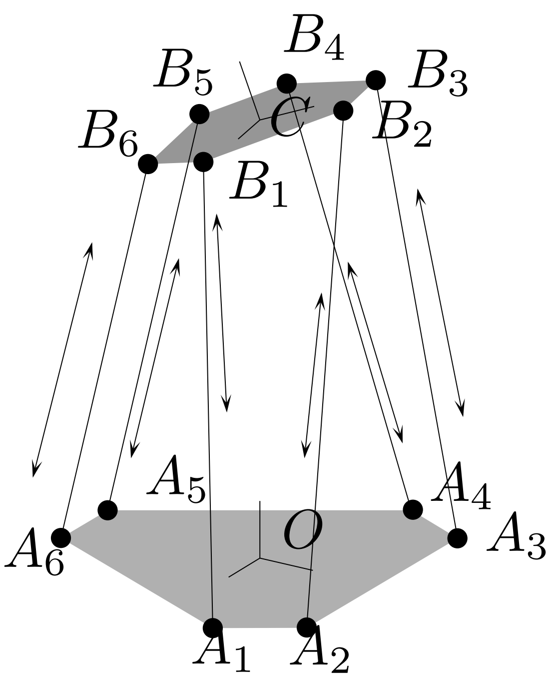

- *output*: the pose of the platform that may be parametrized by a set of parameters $\mathbf{X}$
- *input*: the lengths $\rho$ of the legs

Kinematics: the relation between $\mathbf{X}$ and $\rho$

# Kinematics

- *Inverse kinematics*: $\mathbf{X} \rightarrow \rho$
- *Direct kinematics*: $\rho \rightarrow \mathbf{X}$

Inverse kinematics:




- coordinates of the $A_i$ are known
- coordinates of $C$ are known
- $R$: rotation matrix
- $\mathbf{CB} = R\mathbf{CB_r}$
- $\mathbf{AB} = \mathbf{AO} + \mathbf{OC} + \mathbf{CB}$
- $||\mathbf{AB}||^2 = \rho^2$

# Direct kinematics

Constraints: $||\mathbf{AB}(\mathbf{X})||^2 = F(\mathbf{X}) = \rho^2$

- a square system of non-algebraic or non-algebraic equations
- admits, in general, a finite number of solutions: up to 40 for the Gough platform
- efficient algorithms for finding all solutions: Groebner basis, interval analysis, elimination
- 20 years of work

**Note**: direction of the leg (aka bearing) may be used as additional constraints for the DK

# Direct kinematics

*admits, in general, a finite number of solutions*

$$\Downarrow$$

for fixed values of $\rho$ there are no motion of the platform

in general: assume that you have found a solution $\mathbf{X_0}$, then there is no other solution in the neighborhood of $\mathbf{X_0}$

**Rank theorem**: if $\mathbf{J^{-1}} = \partial F/\partial \boldsymbol{X}$ has full rank, then there is no other solution in the neighborhood of $\boldsymbol{X}_0$

# Direct kinematics

Infinite rigidity unless $\mathbf{J^{-1}}$ is singular

If $\mathbf{J^{-1}}$ is singular

$$\Downarrow$$

- platform will move although the leg lengths are fixed
- loss of rigidity
- loss of controlability
- the infinitesimal motion of the platform may lead to another singularity $\Rightarrow$ finite motion

# Variationnal analysis

Time-derivative of the previous system

$$\dot{\rho} = \mathbf{J}^{-1}\dot{\mathbf{X}} \qquad \Delta\rho = \mathbf{J}^{-1}\Delta\mathbf{X}$$

This is not velocity relation as $\dot{\mathbf{X}}$ is not a representative of the angular velocity of the platform

if $\mathbf{J}^{-1}$ is singular there are $\dot{\mathbf{X}} \neq \mathbf{0}$ such that $\dot{\rho} = \mathbf{0}$

$$\Downarrow$$

- infinitesimal motion of the platform for fixed leg lengths
- loss of control and rigidity
- close to where $J^{-1}$ is singular: large amplification factor between the change in leg lengths and the amplitude of the platform motion

# Statics analysis

- $\tau$: force in the leg (directed along the vector $\mathbf{AB}$)
- $\mathcal{F}, \mathcal{M}$: forces/torques acting on the platform

Mechanical equilibrium:

- $\sum \tau_i \frac{\mathbf{A_i B_i}}{\rho_i} = (\mathcal{F}_x, \mathcal{F}_y, \mathcal{F}_z)^T$
- $\sum \tau_i \frac{\mathbf{CB_i} \times \mathbf{A_i B_i}}{\rho_i} = (\mathcal{M}_x, \mathcal{M}_y, \mathcal{M}_z)^T$

In matrix form:

$$(\mathcal{F}, \mathcal{M})^T = \mathbf{U}(\mathbf{X})^T \tau$$

with the i-th row $U_i$ of $\mathbf{U}$:

$$U_i = (\frac{\mathbf{A_i B_i}}{\rho_i}, \frac{\mathbf{CB_i} \times \mathbf{A_i B_i}}{\rho_i})$$

# Statics analysis

$$(\mathcal{F}, \mathcal{M})^T = \mathbf{U}(\mathbf{X})^T \tau$$

for a given pose $\mathbf{X}$ we have here a linear system in the $\tau$

$$\Downarrow$$

$$\tau_i = \frac{|\mathbf{M_i}|}{|\mathbf{U}|}$$

everything is fine unless $|\mathbf{U}| = 0 \Rightarrow \tau_i \rightarrow \infty$

# Statics analysis

$$\mathbf{U}(\mathbf{X}) = \mathbf{H}(\mathbf{X})\mathbf{J}^{-1}(\mathbf{X})$$

where $\mathbf{H}$ is a non-singular matrix that is dependent only on the choice of the parameters for representing the orientation of the platform

Hence: $\mathbf{U}$ singular $\Longleftrightarrow \mathbf{J}^{-1}$ singular

| |
|---|
| Poses $\mathbf{X}$ where $\mathbf{U}$ is singular are called singular poses of the parallel robot |

# Singularity analysis

Why studying singularities ?

Drawbacks:

- loss of control
- possible breakdown of the robot due to large forces in the leg

Advantages

- high sensitivity of the $\tau$ with respect to $\mathcal{F}, \mathcal{M}$: force/torque sensor

# Singularity analysis

What is the problem ?

- we have a matrix $\mathbf{U}(\mathbf{X})$ in analytical form
- singularity are obtained for $\mathbf{X}$ such that $|\mathbf{U}(\mathbf{X})| = 0$

$$\Downarrow$$

just compute $|\mathbf{U}(\mathbf{X})| = 0$ and solve $\ldots$

but

- $|\mathbf{U}(\mathbf{X})|$ is a huge expression
- and solving if in $\mathbf{X}$ is not a trivial task

We need tools and methodologies

# Singularity analysis

Historical background

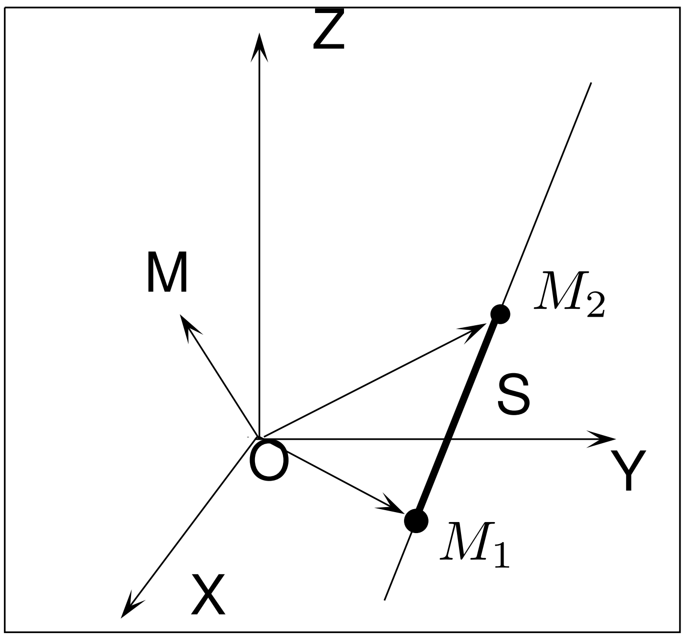
- In 1645, Sir Christopher Wren shows that curved surface may be built with line generators
- In 1813 Cauchy shows that the singularity of an articulated octahedron could be obtained only for concave configurations
- In 1908 the subject of the Prix Vaillant from the Academy of Sciences was to determine under which conditions a parallel mechanism may exhibit finite motion (prize won by Borel and Bricard)

# Plücker vectors

Let $\mathcal{L}$ be a line in space and select two points $M_1, M_2$ on this line

Let $\mathbf{P}$ be the 6-dimensional vector

$$\mathbf{P} = \begin{pmatrix} \frac{\mathbf{M_1 M_2}}{||\mathbf{M_1 M_2}||} \\ \frac{\mathbf{OM_1} \times \mathbf{OM_2}}{||\mathbf{M_1 M_2}||} \end{pmatrix} = \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}$$



# Plücker vectors

$\mathbf{P}$ is the normalized Plücker vector of the line $\mathcal{L}$

- a line has a unique normalized Plücker vector
- let $\mathbf{P_1}, \mathbf{P_2}$ be the Plücker vectors of two lines $\mathcal{L}_1, \mathcal{L}_2$. These two lines will meet iff $\mathbf{p_1}.\mathbf{q_2} + \mathbf{q_1}.\mathbf{p_2} = 0$

Remember

$$U_i = (\frac{\mathbf{A_iB_i}}{\rho_i}, \frac{\mathbf{CB_i} \times \mathbf{A_iB_i}}{\rho_i})$$

$U_i$ is the Plücker vector of leg $i$

if $\mathbf{U}$ is singular, then we have a linear dependency between the Plücker vectors of the legs

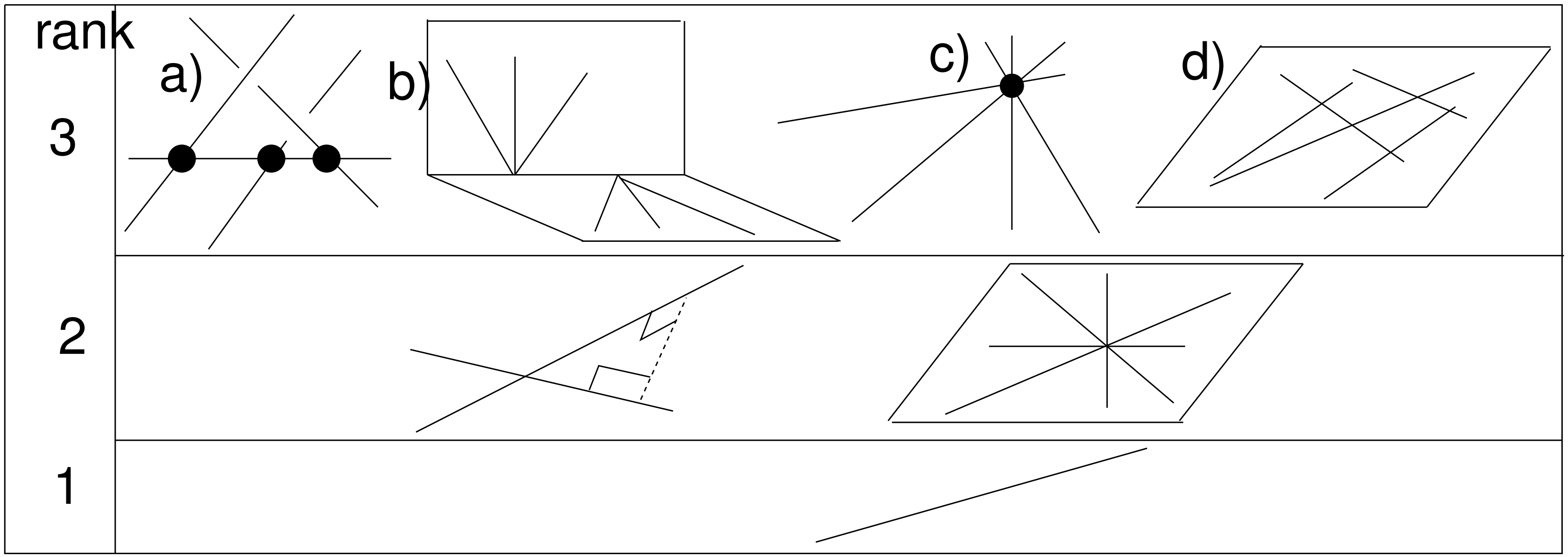
# Grassmann geometry

- the rank of a variety spanned by a set of $n$ Plücker vectors if the rank of $\mathbf{U}$
- the rank cannot be greater than 6
- if $\mathbf{U}$ is singular then the rank will be lower than 6

A singularity will occur only for certain geometrical configurations of the robot that may be determined using Grassmann geometry

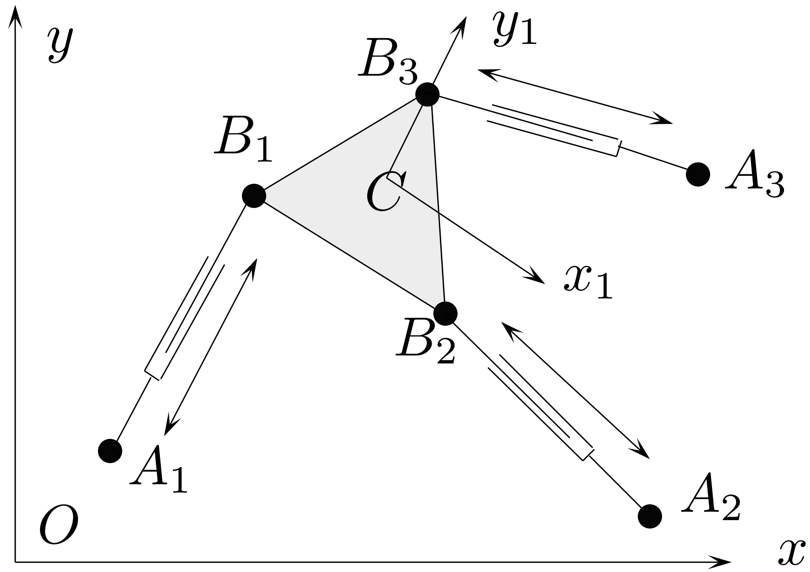
# Grassmann geometry

Variety with rank from 1 to 3



# Grassmann geometry

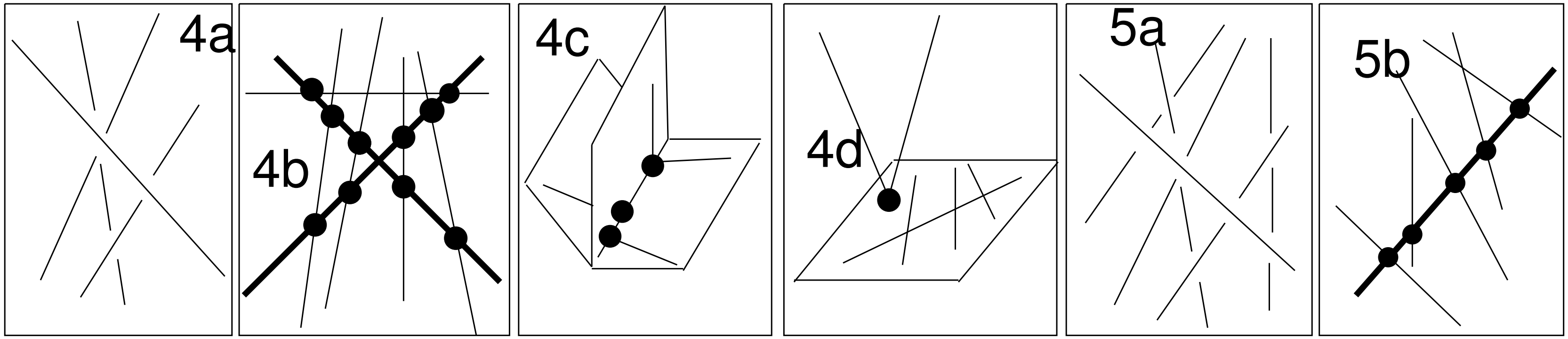
Use example




- three Plücker vectors $\boldsymbol{P_1}, \boldsymbol{P_2}, \boldsymbol{P_3}$ span a variety of dimension $< 3 \Rightarrow$ singularity
- Grassmann: this occurs only if the lines lie in the same plane and meet at a common point
- writing this condition as function of $\boldsymbol{X}$ gives the only singularity condition

# Grassmann geometry

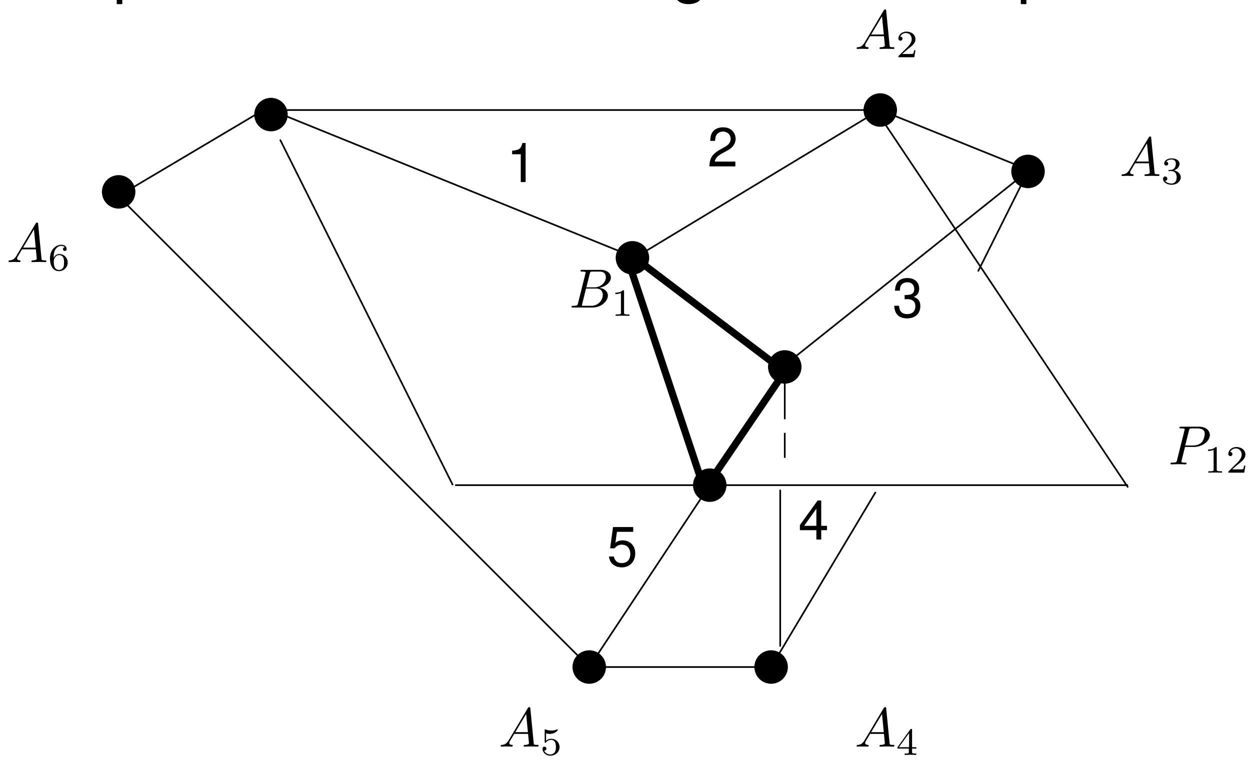
Variety with rank from 4 to 5




- Grassmann geometry gives a set of geometrical condition for which a singularity will occur
- each such condition can be expressed as function of $\mathbf{X}$ (factorization of $|\mathbf{J}^{-1}|$)
- plugging in the singularity condition in $\mathbf{J}^{-1}$ and looking at the kernel of this matrix provides the nature of the infinitesimal motion

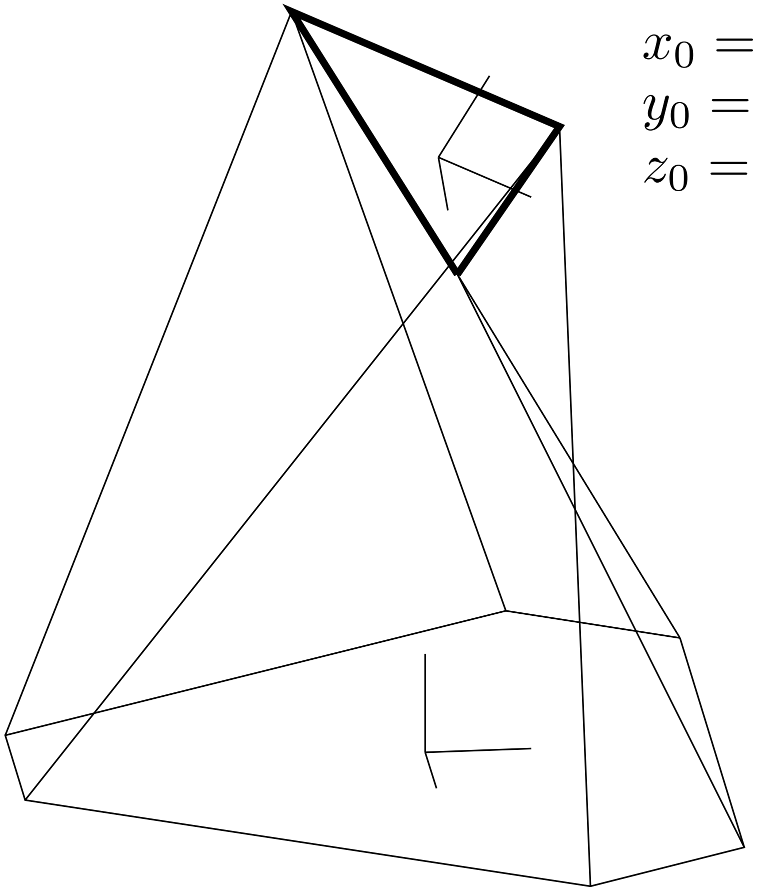
# Grassmann geometry

All lines in a plane or intersecting the same point in this plane

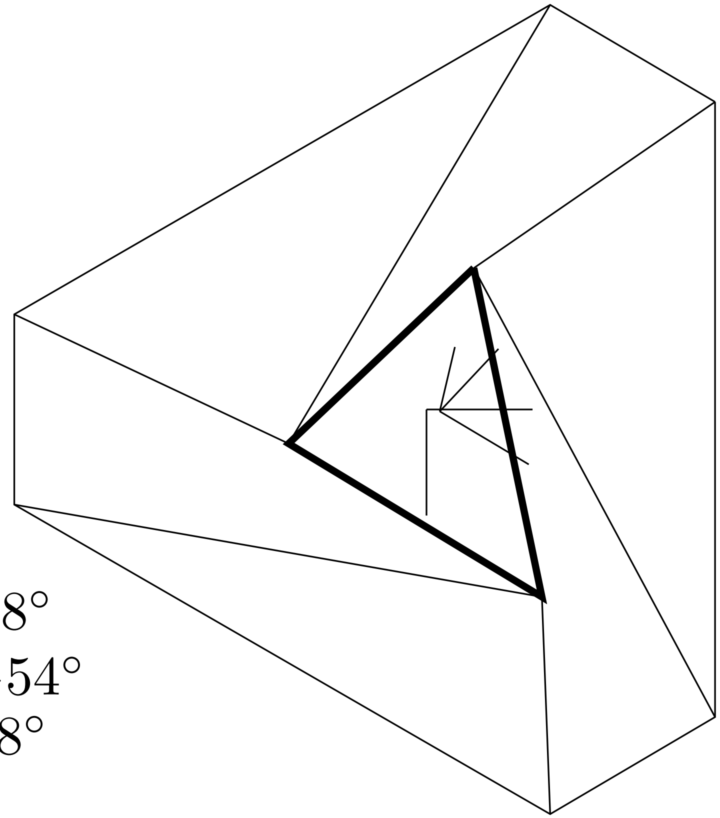


# Grassmann geometry

All lines intersecting the same line



$$x_0 = 0.0835$$
$$y_0 = 0.437$$
$$z_0 = 20$$



$$\psi = 48^\circ$$
$$\theta = -54^\circ$$
$$\phi = 18^\circ$$

# Tools: singularity detection

A very important practical problem

- determine if a singularity exists within a given workspace for $\mathbf{X}$
- binary answer: yes or no
- if a singularity exists we are not interested in its location

Classical approach

- use optimization (e.g. find $\mathbf{X}$ that minimize $|\mathbf{U}(\mathbf{X})|^2$
- unsafe, slow

# Tools: singularity detection

To solve efficiently this problem we will use interval analysis

Interval $\mathcal{X} = [\underline{x}, \overline{x}]$

Assume that we have

- a function $F(x)$
- a range $\mathcal{X}$ for $x$

**interval evaluation** of $F$ when $x \in \mathcal{X}$: a range $[A, B]$ such that

$$\forall\, x \;\in\; \mathcal{X} \;\; \text{we have} \;\; : A \leq F(x) \leq B$$

# Tools: singularity detection

How to construct an interval evaluation ? the simplest one is the **natural evaluation**: substitute each mathematical operator by its interval equivalent

**Example**: $f(x) = x + \sin(x)$, $x \in [1.1, 2]$

$$\begin{aligned} F([1.1,2]) &= [1.1,2] + \sin([1.1,2]) \\ &= [1.1,2] + [0.8912,1] = [1.9912,3] = [A,B] \end{aligned}$$

# Tools: singularity detection

**Properties**

- interval equivalent for any mathematical operators: no limitation to algebraic equations
- can be implemented to take into account round-off errors: numerically robust
- if $0 \not\in [\underline{F}, \overline{F}]$, then there is no $x$ in $\mathcal{X}$ that cancel $F$
- extrema of $F$ are bounded by the values of $A, B$

# Tools: singularity detection

Bounded workspace: $\mathcal{W}$ (defined by a set of intervals for $\mathbf{X}$, defining a box in a $m$ dimensional space)

Ingredients:

- choose an arbitrary point $\mathbf{X_0}$ in $\mathcal{W}$, compute $|\mathbf{U}(\mathbf{X_0})|$ with interval arithmetic, determine its sign (say $> 0$)
- $L$: a list of $n$ boxes $\mathcal{B}_j$ with initially $n = 1, L = \{\mathcal{W}\}$

If we find a pose $\mathbf{X_1}$ such that $|\mathbf{U}(\mathbf{X_1})| < 0$, then $\mathcal{W}$ includes at least one singularity

# Tools: singularity detection

**Algorithm** set $j = 1$

1. if $j > n$, then exit, **NO SINGULARITY**
2. compute the interval evaluation of $|\mathbf{U}(\mathcal{B}_j)| = [A_j, B_j]$
3. if $A_j > 0$, then $j = j + 1$, go to 1
4. if $B_j < 0$, then exit, **SINGULARITY**
5. if $A_j < 0$ and $B_j > 0$, then split $\mathcal{B}_j$ into 2 boxes that are added to $\mathcal{L}$, $j = j + 1, n = n + 2$, go to 1

Very fast algorithm

Note: uncertainties in the geometry of the robot may be taken into account to ensure that the **real** robot is singularity-free

# Methodology: singularity index

Several indexes have been proposed to characterize the closeness to a singularity

- condition number of $\mathbf{U}$
- dexterity
- ...

But all of them have drawbacks and none of them have a physical meaning

# Methodology: singularity index

Another approach: investigate the workspace of the robot such that all $\tau$ satisfy $|\tau_i| \leq \tau_{max}$

Finding this workspace:

- analytically for simple robot
- interval analysis for more complex robot

# Methodology: singularity index

2 dof simple robot

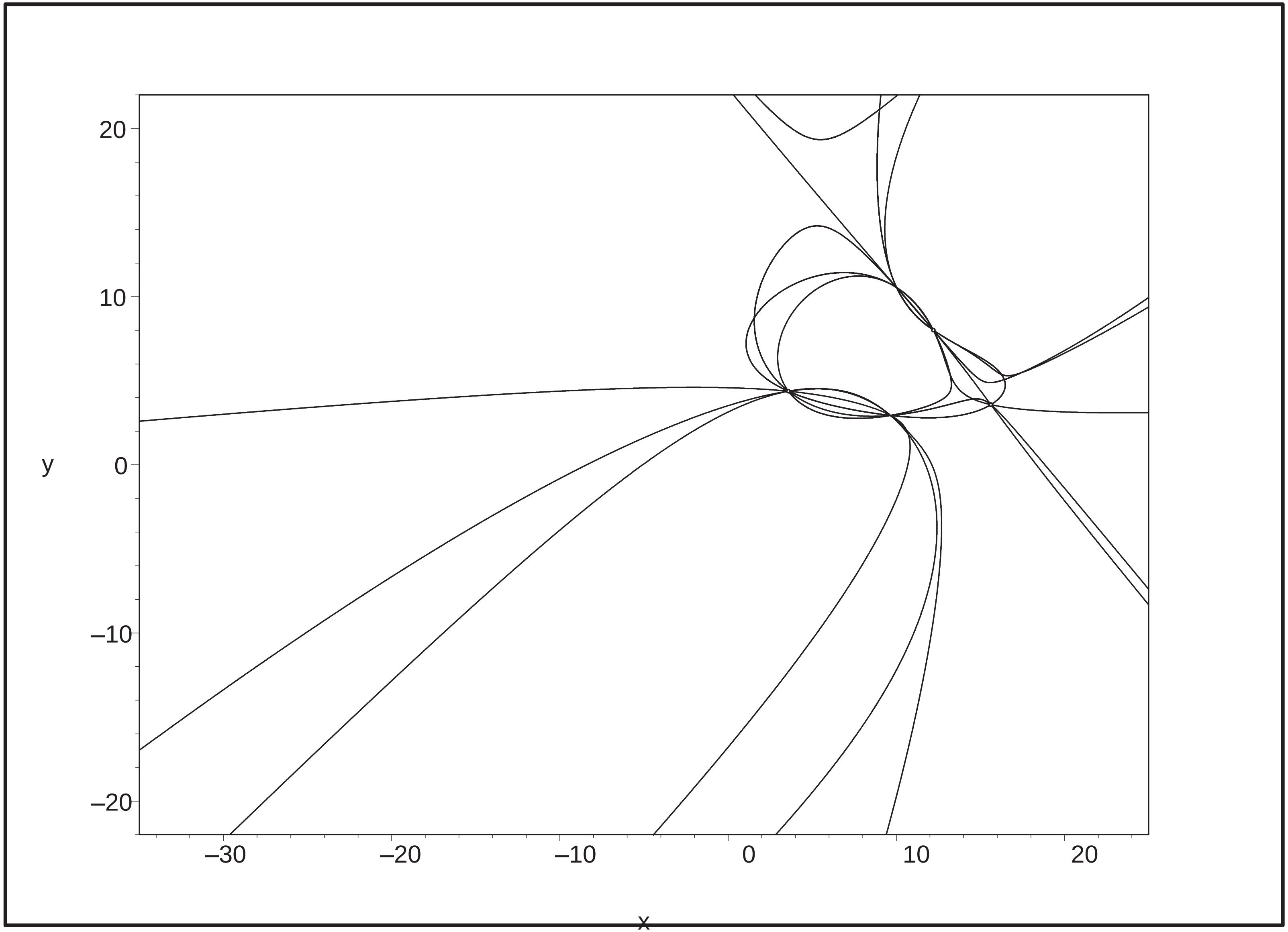
$$\tau_i = \frac{|\mathbf{M_i}|}{|\mathbf{U}|}$$

On the border of the workspace we have for at least one leg

- $\tau_{max}|\mathbf{U}| = |\mathbf{M_i}|$
- or $\tau_{max}|\mathbf{U}| = |\mathbf{M_i}|$

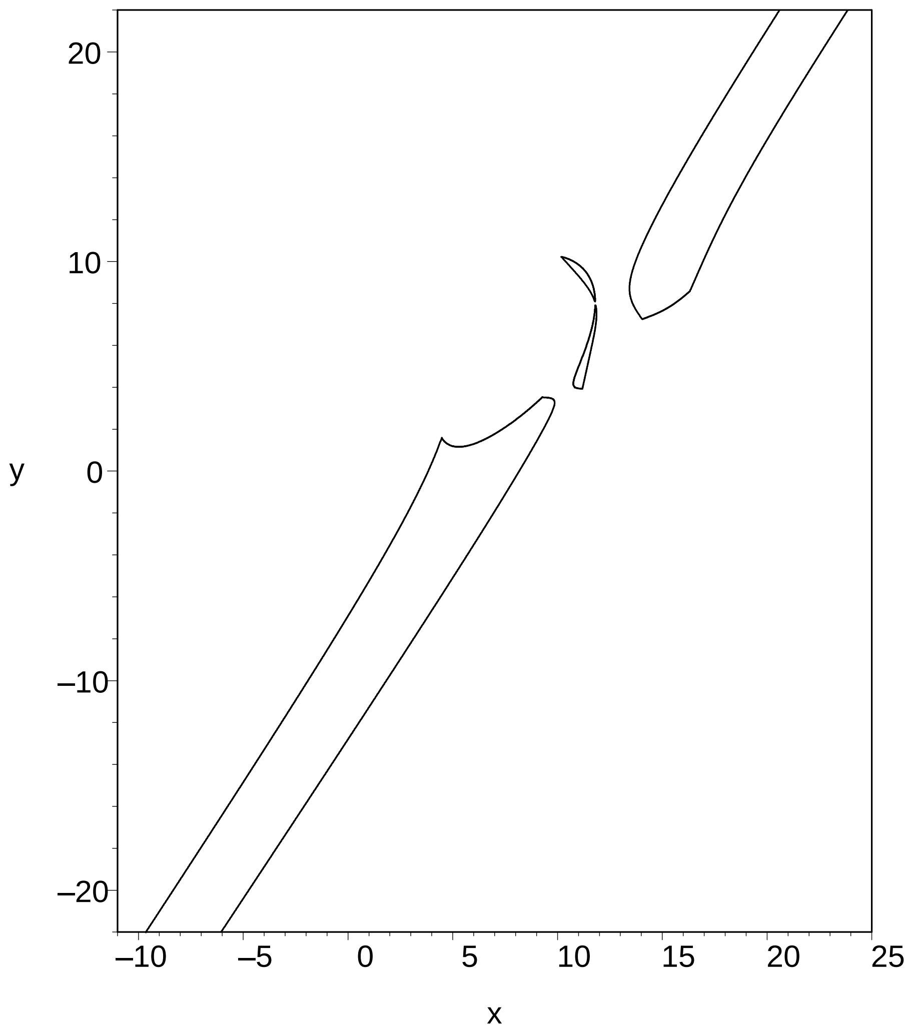
These equations define curves in the plane

# Methodology: singularity index



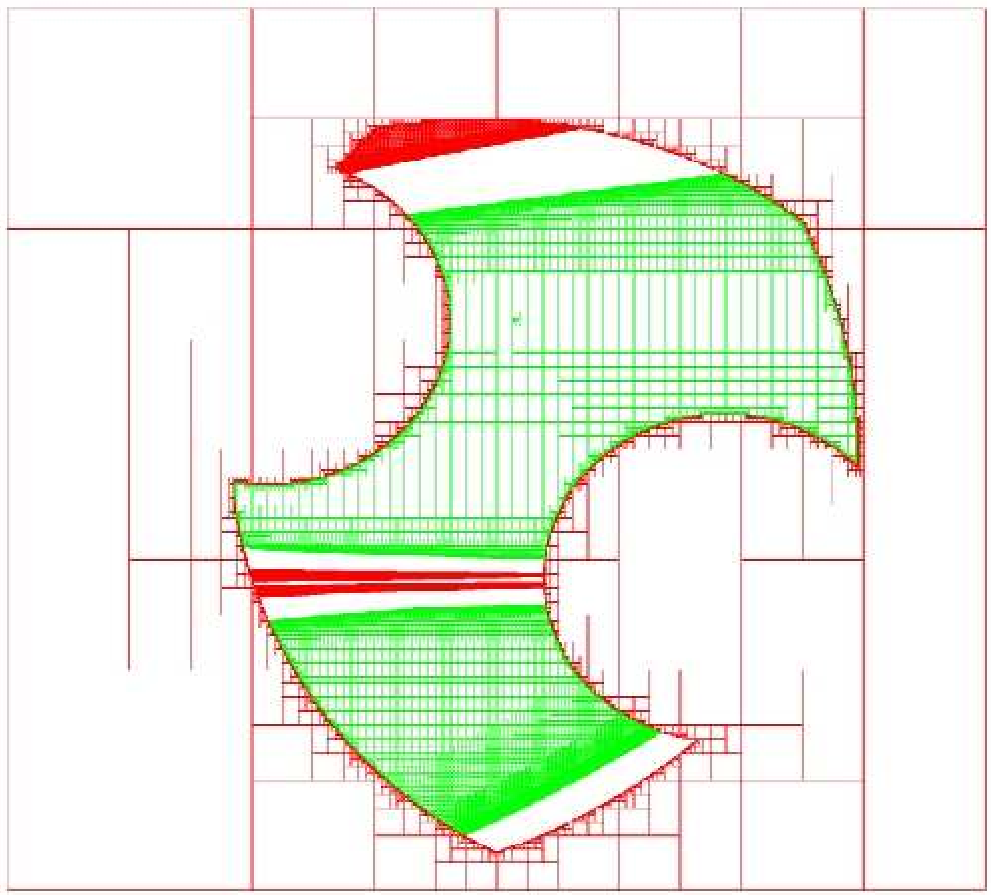
# Methodology: singularity index

A simple algorithm allows one to find the workspace border



# Methodology: singularity index

Complex robot: cross-section for a 6 dof robot by using interval analysis



# Conclusion

- singularity are relatively well mastered for parallel robot in terms of analysis for a given architecture
- analysis may take into account uncertainties in the geometry
- weaker for general analysis of mechanical architectures
- prospective:
  - bearing may be measured to simplify the direct kinematics: but measured with uncertainties
  - cable instead of rigid leg: they can only pull
  - cable with deformation (elasticity, sagging)